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Arithmetic Fuchsian groups with signature (1; e)

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§ 1. Introduction.

In the previous papers [17], [18] we determined all arithmetic triangle Fuchsian groups. The purpose of this paper is to determine all arithmetic Fuchsian groups with signature $(1; e)$. In § 2, we prove that for arbitrary non-negative integers g and t there exist finitely many arithmetic Fuchsian groups with signature $(g; e_1, e_2, \cdots, e_t)$ up to $SL_2(\boldsymbol{R})$-conjugation (Theorem 2.1). In § 3 we deal with arithmetic Fuchsian groups Γ with signature $(1; e)$ (i.e. $g=1$, $t=1$). We give a necessary and sufficient condition for such a group Γ to be arithmetic. More precisely, assume that Γ contains -1_2. Then Γ has the following presentation:

$$\Gamma=\langle\alpha, \beta, \gamma \mid \alpha\beta\alpha^{-1}\beta^{-1}\gamma=-1_2, \gamma^e=-1_2\rangle,$$

where α and β are hyperbolic elements of $SL_2(\boldsymbol{R})$ and γ is an elliptic (resp. a parabolic) element such that $\operatorname{tr}(\gamma)=2\cos(\pi/e)$. Among such triples (α, β, γ) of generators of Γ we can find a certain fundamental triple $(\alpha_0, \beta_0, \gamma_0)$. Let $x=\operatorname{tr}(\alpha_0)$, $y=\operatorname{tr}(\beta_0)$, $z=\operatorname{tr}(\alpha_0\beta_0)$. Then the condition for Γ to be arithmetic can be expressed in terms of x, y, z. We can also obtain an explicit expression of the quaternion algebra associated with Γ (Theorem 3.4). In § 4 using Theorem 3.4 of § 3 we determine all arithmetic Fuchsian groups with signature $(1; e)$ and list them up (Theorem 4.1). In Fricke-Klein [7] we can find some examples of arithmetic Fuchsian groups with signature $(1; e)$.

§ 2. Arithmetic Fuchsian groups.

We recall the definition of arithmetic Fuchsian groups. Let k be a totally real algebraic number field of degree n. Then we have n distinct $\boldsymbol{Q}$-embeddings φ_i $(1\leqq i\leqq n)$ of k into the real number field $\boldsymbol{R}$, where φ_1 is the identity. Let A be a quaternion algebra over k which is unramified at the place φ_1 and ramified at all other infinite places φ_i $(2\leqq i\leqq n)$. Then there exists an $\boldsymbol{R}$-isomorphism

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