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Invariant subspaces of unitary operators

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§ 1. Introduction.

Let $\mathcal{H}$ be a separable, complex Hilbert space and U a unitary operator on $\mathcal{H}$. If

$$\mathcal{H}=M(C)=\cdots\oplus U^{*}C\oplus C\oplus UC\oplus U^{2}C\oplus\cdots$$

for some closed subspace C of $\mathcal{H}$, then U is said to be a bilateral shift of multiplicity n, where U^{*} is the adjoint operator of U and n is the dimension of C. When U is an isometry with $\mathcal{H}=M_{+}(C)=C\oplus UC\oplus U^{2}C\oplus\cdots$, then U is said to be a unilateral shift of multiplicity n. The study of invariant subspaces of shifts was begun by Beurling [**1**]. He characterized all invariant subspaces of unilateral shifts of multiplicity one. Lax [**8**] extended Beurling's result to unilateral shifts of finite multiplicity. Helson-Lowdenslager [**6**] and Halmos [**5**] characterized all invariant subspaces of bilateral shifts of arbitrary multiplicity. We are going to study invariant subspaces of unitary operators in order to generalize previous results concerning bilateral shifts.

A unitary operator U on $\mathcal{H}$ is said to be pure if every invariant subspace for U is reducing. Every unitary operator U can be written as the direct sum of a bilateral shift and a pure unitary operator [**2**; pp. 62-63]. Namely, there exists a closed subspace C of $\mathcal{H}$ such that $U^{n}C$, $n=0, \pm 1, \pm 2, \cdots$ are mutually orthogonal and the restriction of U to $\mathcal{H}\ominus M(C)$ is pure. Setting $\mathcal{K}=\mathcal{H}\ominus M(C)$, we get a decomposition

$$(*)\qquad \mathcal{H}=M(C)\oplus\mathcal{K}\,, \qquad U=S\oplus V\,,$$

so that $S=U|M(C)$ is a bilateral shift and $V|\mathcal{K}$ is pure. It is known that such decomposition is not necessarily unique. In fact, any bilateral shift of infinite multiplicity allows infinitely many non-isomorphic decompositions of the form (*). We call a decomposition $\mathcal{H}=M(C_{0})\oplus\mathcal{K}_{0}$ of the form (*) maximal if we have $M(C')=M(C_{0})$ for any decomposition $\mathcal{H}=M(C')\oplus\mathcal{K}'$ of the form (*) with

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