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On 3-manifolds admitting orientation-reversing involutions

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1. Statement of main results.

Throughout this paper, spaces and maps will be considered in the piecewise-linear category, unless otherwise specified. The purpose of this paper is to discuss some properties of a pair (M, α), where M is a closed, oriented 3-manifold, and α is an orientation-reversing involution on M (that is, $\alpha^2=$ identity, and $\alpha_*[M]=-[M]$ for the fundamental class $[M]$ of M).

The following is perhaps known, but no reference could be found.

THEOREM I. *Given a pair (M, α), then the torsion subgroup $T_1(M; Z)$ of the homology group $H_1(M; Z)$ is isomorphic to a direct double $A \oplus A$ or a direct sum $A \oplus A \oplus Z_2$ for some A.*

For example, the lens space $L(p, q)$, $p>2$, does not admit any orientation-reversing involution, though the projective 3-space $P^3=L(2, 1)$ admits a unique orientation-reversing involution α, whose fixed point set $\mathrm{Fix}(\alpha, P^3)$ is the topological sum P^0+P^2 of the projective 0-space P^0 (=one point) and the projective 2-space P^2. (Cf. K. W. Kwun [**15**].)

By $\mathfrak{C}$ we denote the class of finitely generated abelian groups with torsion parts of the form $A \oplus A$ or $A \oplus A \oplus Z_2$.

DEFINITION 1.1. For any $G \in \mathfrak{C}$, we define $\sigma(G)$ to be 0 or 1, according to whether the torsion subgroup of G is a direct double or not. By using Theorem I, we define $\sigma(M)=\sigma(H_1(M; Z))$ for any pair (M, α).

The following shows enough that the homological classification of Theorem I is complete, where a 3-manifold is irreducible if any imbedded 2-sphere bounds a 3-ball in it.

THEOREM II. *For any $G \in \mathfrak{C}$ there exists a pair (M, α) with $H_1(M; Z)=G$ so that if $\sigma(G)=0$, then M is connected and irreducible, or if $\sigma(G)=1$, then $M=M_1 \# P^3$ with M_1 connected and irreducible, and α preserves the factors.*

Some G with $\sigma(G)=1$ is probably still realizable by a pair (M, α) with M connected and irreducible, but the following may be noted:

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