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On the existence of harmonic functions in L^p

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Let D be a domain in the n-dimensional Euclidean space R^n ($n\geqq 2$), and let $A_p(D)$ (resp. $H_p(D)$), $1<p<\infty$, be the space of all functions in $L^p(D)$ each of which is holomorphic (resp. harmonic) in D if $n=2$ (resp. $n\geqq 3$). Carleson [2] proved in case $n=2$ that

i) if $p>2$ and $C_q(R^2-D)>0$, $1/p+1/q=1$, then $A_p(D)$ contains a non-constant function;

ii) if $p>2$ and $\Lambda_{2-q}(R^2-D)<\infty$, then $A_p(D)=\{0\}$. Here C_α denotes the Riesz capacity with respect to the kernel $r^{\alpha-n}$, and Λ_α denotes the α-dimensional Hausdorff measure.

To improve this result, it is convenient to use the Bessel capacity; the Bessel capacity of index (α, r), $\alpha>0$, $1<r<\infty$, is denoted by $B_{\alpha,r}$ (cf. Meyers [4]). Further, we say that a class of functions is non-trivial if it contains a non-constant function.

Our main aim is to prove the following theorems.

THEOREM 1. (i) *If* $B_{1,q}(R^2-D)=0$, *then* $A_p(D)=\{0\}$.

(ii) *If* $p\geqq 2$ *and* $B_{1,q}(R^2-D)>0$, *then* $A_p(D)$ *is non-trivial.*

(iii) *If* $p<2$ *and* R^2-D *contains at least two points, then* $A_p(D)$ *is non-trivial.*

THEOREM 2. (i) *If* $B_{2,q}(R^n-D)=0$, *then* $H_p(D)=\{0\}$.

(ii) *If* $2q\leqq n$ *and* $B_{2,q}(R^n-D)>0$, *then* $H_p(D)$ *is non-trivial.*

(iii) *If* $2q>n$, $q\neq n$ *and* R^n-D *contains at least two points, then* $H_p(D)$ *is non-trivial.*

(iv) *If* $q=n$ *and* $R^n-D\supset\{x^0, 0, -x^0\}$, $x^0\neq 0$, *then* $H_p(D)$ *is non-trivial.*

REMARK 1. (i) If $q<n<2q$ and $D=R^n-\{x^{(1)}, x^{(2)}\}$, $x^{(1)}\neq x^{(2)}$, then $H_p(D)=\{cu\,;\,c\in R^1\}$, where

$$u(x)=|x-x^{(1)}|^{2-n}-|x-x^{(2)}|^{2-n}.$$

(ii) If $q>n$ and $D=R^n-\{x^{(1)}, x^{(2)}\}$, $x^{(1)}\neq x^{(2)}$, then $H_p(D)=\left\{\sum_{i=0}^{n} c_i u_i\,;\, c_i\in R^1 \text{ for } i=0, 1, \cdots, n\right\}$, where

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