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LA-groups

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The number of automorphisms of a finite p-group G has been an interesting subject of research for a long time. It has been conjectured that, if G is a non-cyclic finite p-group of order p^n, $n>2$, then the order of G divides the order of the group of automorphisms of G. This has been established for abelian p-groups and for certain classes of finite p-groups. In this paper we show that the conjecture is also true for some other classes of non-abelian p-groups.

A finite p-group G, which satisfies the above conjecture, is called an LA-group.

Throughout this paper G stands for a finite non-abelian group of order p^n (p a prime number), commutator subgroup G' and center Z. We denote the order of any group H by $|H|$. Also we take the lower and the upper central series of a finite p-group G to be:

$$G=L_0>L_1=G'>\cdots>L_c=1$$

and

$$1=Z_0<Z_1=Z<\cdots<Z_c=G,$$

where c is the class of G. For $c=2$, G is an LA-group ([3]). So we shall assume that $c>2$. The invariants of G/L_1 are taken to be $m_1\geqq m_2\geqq\cdots\geqq m_t\geqq 1$ and $|G/L_1|=p^m$. Similarly we take the invariants of Z to be $k_1\geqq k_2\geqq\cdots\geqq k_s\geqq 1$ and $|Z|=p^k$. We denote by $A(G)$, $I(G)$, $A_c(G)$ the groups of automorphisms, inner automorphisms, central automorphisms of G respectively. $\mathrm{Hom}(G, Z)$ is the group of homomorphisms of G into Z. Finally $P(G)=\langle x^p \mid x\in G\rangle$ and $E(G)=\langle x\in G \mid x^p=1\rangle$.

G is called a PN-group, if G has no non-trivial abelian direct factor.

We begin with:

LEMMA 1. (i) *If G is a PN-group, $|A_c(G)|=p^a$, where*

$$a=\sum \min(m_j, k_i).$$

(ii) *If $G=H\times K$, where H is abelian and non-trivial and K is a PN-group, then*

$$|A_c(G)|=|A(H)|\,|A_c(K)|\,|\mathrm{Hom}(K, H)|\,|\mathrm{Hom}(H, Z(K))|.$$