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The dimension of the space of cusp forms on the Siegel upper half plane of degree two related to a quaternion unitary group

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§ 0. Introduction.

0.1. The main purpose of this paper is to give an explicit calculation of the dimension of the spaces of cusp forms on the Siegel upper half plane of degree two with respect to some arithmetic discontinuous groups having zero-dimensional cusps. Such groups are defined from A-hermitian forms of degree two, where A is an indefinite division quaternion algebra over the rational number field $\boldsymbol{Q}$. The same result was obtained by H. Yamaguchi [18] by quite a different method; while Yamaguchi uses the Hirzebruch-Riemann-Roch theorem, our calculation is based on the Selberg trace formula.

Let G be the A-unitary group of degree two. Since A is indefinite, this determines a linear algebraic group G over $\boldsymbol{Q}$ up to $\boldsymbol{Q}$-isomorphisms. Denote by $a \rightarrow a'$ ($a \in A$) the canonical involution of A and by $\mathfrak{O}$ a maximal order of A, and let $M_2(A)$ denote the total matrix algebra of degree two over A. As an explicit presentation of G, we define the group of $\boldsymbol{Q}$-rational points as

$$G_{\boldsymbol{Q}}=\left\{S \in M_2(A) \,\middle|\, S\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S'=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\},$$

where $S'=\begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}$ for $S=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A)$. Let N be a natural number. We consider the arithmetic discontinuous group $\Gamma(N)$ of $G_{\boldsymbol{Q}}$ such that

$$\Gamma(N)=\left\{S=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\boldsymbol{Q}} \,\middle|\, a-1,\, b,\, c,\, d-1 \in N\mathfrak{O}\right\}.$$

In particular, we set $\Gamma=\Gamma(1)$. Since the group $G_{\boldsymbol{R}}$ of $\boldsymbol{R}$-rational points of G is conjugate in $GL(4, \boldsymbol{R})$ to the real symplectic group $Sp(2, \boldsymbol{R})$ of degree two (size 4), so we may identify the arithmetic groups Γ, $\Gamma(N)$ with the discontinuous subgroups of $Sp(2, \boldsymbol{R})$ by a fixed isomorphism of $G_{\boldsymbol{R}}$ to $Sp(2, \boldsymbol{R})$. Denote by $\mathfrak{H}_2$ the Siegel upper half plane of degree two: $\mathfrak{H}_2=\{Z \in M_2(\boldsymbol{C}) \mid {}^tZ=Z,\ \mathrm{Im}(Z)>0\}$. Then, $Sp(2, \boldsymbol{R})$ operates on $\mathfrak{H}_2$ by