J. Math. Soc. Japan
Vol. 33, No. 1, 1981

Some theorems on projective hyperbolicity

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(Received March 28, 1979)

Let M be a manifold with a torsionfree affine connection Γ. For such a Γ, S. Kobayashi has recently introduced in [**Ko 4**] a pseudo-distance p which depends only on the projective structure of Γ, or what is the same thing, on the normal projective connection induced by Γ (cf. [**Ko 2**], Proposition 7.2 or [**K-Na**]). Call Γ, or simply M when Γ is understood, *(complete) projective-hyperbolic* if and only if p is a (complete) metric, i.e. $p(x, y)=0 \Rightarrow x=y$ for all $x, y \in M$. Then Kobayashi and T. Sasaki have proved the following theorems:

(A) ([**Ko 4**]). *If M is a (complete) Riemannian manifold whose Ricci curvature is bounded above by a negative constant, then M is (complete) projective-hyperbolic.*

(B) ([**K-S**]). *If M is a manifold with a complete torsionfree affine connection whose Ricci tensor is positive semi-definite, then p is identically zero.*

These results are parallel to what is known or what is expected to hold for the Kobayashi metric on complex manifolds (for (A) see [**K 1**], Theorem 4.11 on p. 61; for (B) see [**G-W 1**], Conjecture 1a on p. 79). By contrast, we shall prove the following theorems.

THEOREM 1. *Let M be a manifold with a torsionfree affine connection whose Ricci tensor* Ric *is negative semi-definite. Suppose for each maximal geodesic $\gamma: J \rightarrow M$ where J is an open interval in $\boldsymbol{R}$,* $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma})$ *is never identically zero. Then M is projective-hyperbolic.*

THEOREM 2. *Let M be a compact Riemannian manifold with quasi-negative Ricci curvature (i.e. everywhere nonpositive Ricci curvature which is in addition negative in all directions at a point). Then the group of projective transformations is finite.*

Theorem 2 extends the theorems of Couty [**C**] and Kobayashi [**Ko 4**] who proved under the assumption of negative Ricci curvature that this group is discrete and finite, respectively. The analogous fact of Theorem 1 for the Kobayashi metric on complex manifolds is completely false. For instance, there are complete Hermitian metrics on $\boldsymbol{C}$ of negative curvature, yet the Kobayashi

Miller Institute of Basic Research in Science, University of California, Berkeley. Work also supported in part by the National Science Foundation Grant No. MCS 74-23180.