J. Math. Soc. Japan
Vol. 33, No. 1, 1981

Homological coalgebra

By Yukio DOI

(Received Dec. 13, 1978)
(Revised March 5, 1979)

The purpose of this paper is to give an introduction to the homological theory of comodules over coalgebras and Hopf algebras. Section 1 is a self-contained exposition of basic concepts such as cotensor product, injective comodules and change of coalgebras. Some results analogous to the results (Cline-Parshall-Scott [2], Hochschild [5]) on rational modules over affine algebraic groups are proved. Section 2 deals with the representation theory of co-Frobenius coalgebras and coseparable coalgebras. We reproduce in this section some of Lin's results [7] and Larson's results [6], partly with simplified proof. Section 3 deals with the cohomology theory of coalgebras.

Throughout this paper, the field k is fixed. Vector spaces over k are called k-spaces, and linear maps between k-spaces are called k-maps. We freely use the terminology and results of Sweedler [9].

§1. Coalgebras and comodules.

A coalgebra over k is a k-space C together with k-maps $\Delta: C\to C\otimes C$ and $\varepsilon: C\to k$ such that $(I\otimes\Delta)\Delta=(\Delta\otimes I)\Delta$ and $(I\otimes\varepsilon)\Delta=(\varepsilon\otimes I)\Delta=I$. If C is a coalgebra, a left C-comodule is a k-space M together with a k-map $\rho_M: M\to C\otimes M$ such that $(I\otimes\rho_M)\rho_M=(\Delta\otimes I)\rho_M$ and $(\varepsilon\otimes I)\rho_M=I$. If M and N are left C-comodules, a comodule map from M to N is a k-map $f: M\to N$ such that $(I\otimes f)\rho_M=\rho_N f$. The k-space of all comodule maps from M to N is denoted by $\mathrm{Com}_C(M, N)$ and the category of left C-comodules is denoted by ${}^C\mathbf{M}$. Similarly, we define $\mathbf{M}^C$, the category of right C-comodules.

1.1. Cotensor products and injective comodules.

If M is a right C-comodule and N is a left C-comodule, the *cotensor product* $M\square_C N$ is the kernel of the k-map

$$\rho_M\otimes I-I\otimes\rho_N: M\otimes N\to M\otimes C\otimes N.$$

Given comodule maps $f: M\to M'$ and $g: N\to N'$, the k-map $f\otimes g: M\otimes N\to M'\otimes N'$ induces a k-map