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On foliations with the structure group of automorphisms of a geometric structure

By Masaru TAKEUCHI

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Introduction.

In this paper we shall study foliations with the structure pseudogroup Γ of local automorphisms of a certain 2nd order G-structure. Our purpose is to prove a vanishing theorem for certain characteristic classes of such Γ-foliations and to give a geometric construction of examples of these Γ-foliations.

Let G/U be a semi-simple flat homogeneous space of $\dim G/U=q$ in the sense of Ochiai [12]. It is a connected homogeneous space, on which a semi-simple Lie group G acts transitively and effectively, and $\mathfrak{g}=\mathrm{Lie}\, G$, the Lie algebra of G, has a graded Lie algebra structure:

$$\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1, \qquad \dim \mathfrak{g}_{-1}=q,$$

with $\mathfrak{u}=\mathrm{Lie}\, U=\mathfrak{g}_0+\mathfrak{g}_1$. We identify $\mathfrak{g}_{-1}$ with $\boldsymbol{R}^q$ by a basis for $\mathfrak{g}_{-1}$, and then $\boldsymbol{R}^q$ with an open neighbourhood of the origin U in G/U by the imbedding $\mathfrak{g}_{-1}\ni x\mapsto(\exp x)U\in G/U$. Then we can define an imbedding ι of G into the 2nd order frame bundle $P^2(G/U)$ of G/U by

$$\iota(a)=j_0^2(a) \qquad \text{for} \quad a\in G.$$

In particular, ι identifies U with a Lie subgroup of the structure group $G^2(q)$ of $P^2(G/U)$. Let B be a smooth manifold of $\dim B=q$. A U-subbundle Q of the 2nd order frame bundle $P^2(B)$ of B is called a *2nd order structure of type G/U* over B. For instance, the image $Q_G=\iota(G)$ of ι is a 2nd order structure of type G/U over G/U. Let $\Gamma=\Gamma(Q)$ denote the pseudogroup of all local diffeomorphisms φ of B such that the 2nd prolongation $\varphi^{(2)}$ leaves Q invariant. We shall study Γ-foliations for these pseudogroups Γ.

For example, the pseudogroup Γ of local projective or conformal transformations for a Riemannian metric on a smooth manifold B is obtained in this way from a certain semi-simple flat homogeneous space (cf. §4). The Γ-foliations for these Γ are the so-called projective and conformal foliations.

In general, for a Lie group L and a Lie subalgebra $\mathfrak{h}$ of $\mathrm{Lie}\, L$, we define

$$I_L(\mathfrak{h})=\{f|\mathfrak{h}\,; f \text{ is an } L\text{-invariant polynomial on } \mathrm{Lie}\, L\}.$$