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Finite groups with trivial class groups

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Let A be a finite dimensional semisimple $\boldsymbol{Q}$-algebra and let Λ be a $\boldsymbol{Z}$-order in A. We mean by the class group of Λ the class group defined by using locally free left Λ-modules and denote it by $C(\Lambda)$. Let Ω be a maximal $\boldsymbol{Z}$-order in A containing Λ. We define $D(\Lambda)$ to be the kernel of the natural surjection $C(\Lambda)\rightarrow C(\Omega)$ and $d(\Lambda)$ to be the order of $D(\Lambda)$.

Let G be a finite group and let $\boldsymbol{Z}G$ be the integral group ring of G. Then $\boldsymbol{Z}G$ can be regarded as a $\boldsymbol{Z}$-order in the semisimple $\boldsymbol{Q}$-algebra $\boldsymbol{Q}G$.

In this paper we will try to determine all finite groups G for which $d(\boldsymbol{Z}G)=1$.

Let C_n $(n\geqq 1)$ denote the cyclic group of order n and let D_n $(n\geqq 2)$ denote the dihedral group of order $2n$. Let S_n, A_n denote the symmetric, alternating group on n symbols, respectively.

P. Cassou-Noguès [1] showed that, for a finite abelian group G, $d(\boldsymbol{Z}G)=1$ if and only if $G\cong C_1, C_p$ (p any prime), C_4, C_6, C_8, C_9, C_{10}, C_{14} or $C_2\times C_2$. Hence we have only to treat the nonabelian case.

Our main result is the following:

THEOREM. *A finite nonabelian group G for which $d(\boldsymbol{Z}G)=1$ is isomorphic to one of the groups:* D_n $(n\geqq 3)$, A_4, S_4, A_5.

It is well known (e.g. [14]) that $d(\boldsymbol{Z}A_4)=d(\boldsymbol{Z}S_4)=d(\boldsymbol{Z}A_5)=1$. It is also known that $d(\boldsymbol{Z}D_n)=1$ in each of the following cases: (i) n is an odd prime ([9]); (ii) n is a power of an odd regular prime ([7]); or (iii) n is a power of 2 ([4]). Recently Cassou-Noguès [2] showed that there is an infinite number of pairs (p, q) of distinct odd primes p, q such that $d(\boldsymbol{Z}D_{pq})>1$. It seems difficult to determine all integers n for which $d(\boldsymbol{Z}D_n)=1$.

§1. The group $T(\boldsymbol{Z}G)$.

Let G be a finite group and let (Σ) be the ideal of $\boldsymbol{Z}G$ generated by $\Sigma=\sum_{\sigma\in G}\sigma$. We define the subgroup $T(\boldsymbol{Z}G)$ of $D(\boldsymbol{Z}G)$ to be the kernel of the natural surjection $D(\boldsymbol{Z}G)\rightarrow D(\boldsymbol{Z}G/(\Sigma))$ and $t(\boldsymbol{Z}G)$ to be the order of $T(\boldsymbol{Z}G)$