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A theory of ordinal numbers with Ackermann's schema

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Introduction.

W. Ackermann introduced in [1] a system of axiomatic set theory. A typical character of the system is that the universe V of all sets is a part of the individual domain and besides it is an individual. The system has an interesting axiom schema which generates the sets. It is the following:

$$y_1 \cdots y_n \in V \rightarrow . \forall x[A(x) \rightarrow x \in V] \rightarrow \exists w \in V \forall x[x \in w \leftrightarrow A(x)],$$

where $A(x)$ contains neither the individual constant V nor free variables other than $y_1 \cdots y_n, x$.

In this paper, we modify the above schema to formalize a theory of ordinal numbers and study the strength of the theory. Some theories of ordinal number have been given in Takeuti [6]-[9]. The main purpose of [6]-[9] seems to construct theories of ordinal numbers in which a model of ZF can be constructed. Our interests here are the application itself of Ackermann's schema to the formalization of a theory of ordinal numbers and the degree of the strength of the theory.

The basic logic which we adopt is a second order calculus with the axiom of weak comprehension of the following form:

$$\exists P \forall x_1 \cdots x_n [P x_1 \cdots x_n \leftrightarrow O x_1 \wedge \cdots \wedge O x_n \wedge A],$$

where O is a predicate constant which means "...is an ordinal" and A is any formula. The reason why we weakened the axiom of comprehension in such a form consists of the following two:

1. The author could not estimate the strength of the corresponding theory formalized in the usual second order calculus.

2. Ackermann's set theory has the axiom schema (in the form of Levy and Vaught [4]) $\exists x \forall y [y \in x \leftrightarrow y \in V \wedge A]$ and our weakened axiom of comprehension could be regarded as its natural representation in a second order calculus.