J. Math. Soc. Japan
Vol. 30, No. 3, 1978

Approximation by a sum of polynomials involving primes

By Ming-Chit LIU

(Received April 26, 1976)
(Revised April 17, 1978)

§ 1. Introduction

Based on the Hardy-Littlewood method, Davenport and Heilbronn [3] proved that if $\lambda_1, \cdots, \lambda_s$ are non-zero real numbers, not all of the same sign and not all in rational ratio and if for any given integer $k \geqq 1$, $s \geqq 2^k+1$, then for every $\varepsilon>0$ the inequality

$$|\sum_{j=1}^{s} \lambda_j n_j^k| < \varepsilon$$

has infinitely many solutions in natural numbers n_j. Later, Schwarz [10] showed that if either $s \geqq 2^k+1$ or $s \geqq 2k^2(2 \log k+\log\log k+5/2)-1$ (for $k \geqq 12$), then the inequality

$$|\sum_{j=1}^{s} \lambda_j p_j^k| < \varepsilon \tag{1.1}$$

has infinitely many solutions in prime numbers p_j. A. Baker [1] raised a new kind of approximation by proving that when $k=1$, $s=3$ and A is any arbitrary natural number, the ε in (1.1) can be replaced by $(\log \max p_j)^{-A}$. Recently, Ramachandra [9] has obtained this result for arbitrary k and with the p_j^k replaced by arbitrary integer-valued polynomials $f_j(p_j)$ with positive leading coefficients provided that s satisfies the same condition as that required by Schwarz. In 1974, Vaughan [12] made a remarkable progress in this problem by proving that if $k \geqq 4$,

$$\theta=\begin{cases} 2^{1-k} & \text{if } k \leqq 12\,, \\ (2k^2(2\log k+\log\log k+3))^{-1} & \text{if } k>12\,, \end{cases} \tag{1.2}$$

$$N=[(-\log 2\theta+\log(1-2/k))/(-\log(1-1/k))] \tag{1.3}$$

and

$$s \geqq 2(k+N)+7\,, \tag{1.4}$$

then the ε in (1.1) can be replaced by $(\max p_j)^{-\sigma}$, where σ is any positive constant $<(5(k+1)2^{2(k+1)})^{-1}$. In this paper we shall adapt the elegant method of [12] to establish: