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On an optimal stopping problem and a variational inequality

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A. Bensoussan and J.L. Lions ([1]) has revealed a relation between an optimal stopping problem of an additive functional of a diffusion process and a certain variational inequality. More specifically let $y(t)$ be the solution of the stochastic differential equation:

$$(0.1)\qquad \begin{cases} dy(t)=\sigma(y(t), t)dB_t+g(y(t), t)dt \\ y(t_0)=y_0 . \end{cases}$$

Then they showed that the continuous and strong solution of the following variational inequality (0.2) is the solution of the optimal stopping problem:

$$u(x, s)\equiv E_{x\cdot s}\left[\int_s^{\tau_B^s} e^{-\alpha(t-s)}C(y(t), t)dt+e^{-\alpha(\tau_B^s-s)}D(y(\tau_B^s), \tau_B^s)\right]$$

$$=\inf_{\tau_s} E_{x,s}\left[\int_s^{\tau_s} e^{-\alpha(t-s)}C(y(t), t)+e^{-\alpha(\tau_s-s)}D(y(\tau_s), \tau_s)\right].$$

$$(0.2)\qquad \begin{cases} -\left(\dfrac{\partial u}{\partial t}, v-u\right)+\mathcal{E}_t(u, v-u)+\alpha(u, v-u)\geqq(C, v-u) \\ \qquad\qquad \text{for all} \quad v\in\mathcal{D}[\mathcal{E}_t] \text{ such that } v\leqq D \\ u\in\mathcal{D}[\mathcal{E}_t] \text{ such that } v\leqq D . \end{cases}$$

Here $A(t)$ is the generator of the diffusion process $y(t)$, $\mathcal{E}_t$ is the bilinear form associated with $A(t)$ and $\mathcal{D}[\mathcal{E}_t]$ is the domain of $\mathcal{E}_t$.

However it is in general not easy to show that the (weak) solution of (0.2) is the continuous and strong one, namely, a continuous solution of

$$(0.3)\qquad \begin{cases} -\dfrac{\partial u}{\partial t}+(\alpha-A(t))u-C\leqq 0 \\ \left\{-\dfrac{\partial u}{\partial t}+(\alpha-A(t))u-C\right\}(u-D)=0 \\ u\leqq D . \end{cases}$$