J. Math. Soc. Japan
Vol. 30, No. 1, 1978

Trotter's product formula for nonlinear semigroups generated by the subdifferentials of convex functionals

By Tosio KATO* and Kyûya MASUDA

(Received June 9, 1977)

1. Introduction.

It was proved in [1, 2] that

$$s\text{-}\lim_{n\to\infty}[e^{-(t/n)A_2}e^{-(t/n)A_1}]^n=e^{-tA'}P', \qquad t>0, \tag{1.1}$$

whenever A_1, A_2 are nonnegative selfadjoint operators in a Hilbert space H (with no restriction on their domains). Here P' is the orthogonal projection of H onto the subspace H' spanned by $D'=D(A_1^{1/2})\cap D(A_2^{1/2})$ and A' is the *form sum* of A_1, A_2 (i.e. the selfadjoint operator in H' associated with the densely-defined, closed quadratic form $\|A_1^{1/2}u\|^2+\|A_2^{1/2}u\|^2$).

The purpose of the present paper is to prove a nonlinear analogue of (1.1). As a natural generalization of a nonnegative selfadjoint operator, A_j will be replaced by the subdifferential $\partial\varphi_j$ of a lower semicontinuous, convex function $\varphi_j\not\equiv+\infty$ on H to $]-\infty, +\infty]$; $-\partial\varphi_j$ generates a semigroup $\{e^{-t\partial\varphi_j}\}$ of nonlinear nonexpansive operators on $E_j=cl.D(\varphi_j)$. (For these notions see section 2.) Moreover, we shall admit any finite number N of such semigroups. Thus our result will take the form

$$\lim_{n\to\infty}[e^{-(t/n)\partial\varphi_N}P_N\cdots e^{-(t/n)\partial\varphi_1}P_1]^n x=e^{-t\partial\varphi}x,$$

$$\varphi=\varphi_1+\cdots+\varphi_N, \qquad t\geqq 0,\ x\in cl.D(\varphi), \tag{1.2}$$

where P_j is the nonlinear projection of H onto the closed convex set E_j, and it is assumed that $\varphi\not\equiv+\infty$. Note that $\partial\varphi$ is the analogue of the form sum of the $\partial\varphi_j$. The factors P_j are necessary to ensure that the product on the left of (1.2) makes sense, since $e^{-t\partial\varphi_j}$ is defined only on E_j.

REMARK 1.1. The condition $x\in cl.D(\varphi)$ in (1.2) is a new restriction which was not needed in the linear case (1.1). A straightforward generalization of the latter would be to admit every $x\in H$ and replace x by Px on the right-

* Partially supported by NSF Grant MCS 76-04655.