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Remarks on conditional expectations in von Neumann algebra

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1. Introduction. The conditional expectation has been studied by several authors, e. g. [1] F. Combes, [5] I. Kovács and J. Szüces, [6] M. Nakamura and T. Turumaru and [9] H. Umegaki. Here in this note, we shall make a detailed study on the conditional expectation T_ϕ from M to $(M^{\Sigma\phi})e_\phi$ (See [1]). We then apply it to the strict semi-finiteness of weight.

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2. Conditional expectation. Given a weight ϕ on a von Neumann algebra M, we denote by m_ϕ the *-subalgebra spanned by $n_\phi^* n_\phi$ where $n_\phi=\{x\in M; \phi(x^*x)<+\infty\}$. The linear extension on m_ϕ of $\phi|_{(m_\phi)+}$ will be denoted by $\dot{\phi}$.

The following theorem is a slight modification of [8] Theorem 3, which plays a crucial role in our study. The σ_t-invariance of T follows from the uniqueness of T.

THEOREM 1. *Let M be a von Neumann algebra, ϕ a faithful normal semi-finite weight on M, N a von Neumann subalgebra of M on which $\phi|_{N+}$ is semi-finite.*

Then the following two statements are equivalent;

(i) *N is invariant under the modular automorphism group σ_t associated with ϕ.*

(ii) *There exists a unique σ-weakly continuous conditional expectation T from M on N such that $\phi(x)=\phi\circ T(x)$ for all $x\in M_+$.*

By excluding the condition "$\phi|_{N+}$ is semi-finite" in the above Theorem 1, we get the following proposition.

PROPOSITION 2. *Let M be a von Neumann algebra, ϕ a faithful normal semi-finite weight on M, N a von Neumann subalgebra, e_0 the greatest projection in the σ-weak closure of $m_\phi|_{N+}$.*

Then the following two statements are equivalent;

(i) *e_0Ne_0 is invariant under the modular automorphism group $\Sigma=\{\sigma_t\}$ associated with ϕ.*

(ii) *e_0 is a projection of the subalgebra M^Σ of fixed points of M for Σ and*