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The asymptotic distribution of discrete eigenvalues for Schrödinger operators

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§ 0. Introduction.

We consider the following eigenvalue problem:

$$-\Delta u-p(x)u=\lambda u^{0)}\,, \qquad u\in L^2(R^n)\,.$$

If $p(x)$ does not decay too rapidly at infinity, the above problem has an infinite sequence of negative eigenvalues (bound states) approaching to zero. We denote by $n(r;p)$ $(r>0)$ the number of eigenvalues less than $-r$. In this paper we are concerned with the asymptotic behavior of $n(r;p)$ as r tends to zero.

The asymptotic distribution of negative eigenvalues for the Schödinger operators has been studied in Brownell and Clark [**4**] and McLeod [**6**] under the condition that the potential $p(x)$ is non-negative and sufficiently close to a spherically symmetric potential.

The purpose of the present paper is to study the distribution of eigenvalues by a different method from those in [**4**] and [**6**] without assuming the above condition.

we shall briefly explain our approach. Our method is based on the following proposition (see Birman [**1**]).

PROPOSITION 0.1. *Let H_0 be the unique self-adjoint realization of $-\Delta$ with domain $\mathcal{D}(H_0)=H^2(R^n)$ (the Sobolev space of order 2). Assume that $|p|^{1/2}$ is a $H_0^{1/2}$-compact operator as a multiplicative operator. Let $\mathcal{D}$ be a core of $H_0^{1/2}$. Then, $n(r;p)$ coincides with the maximal dimension of subspaces lying in $\mathcal{D}$ such that*

$$(H_0^{1/2}u,\, H_0^{1/2}u)-(pu,\, u)<-r(u,\, u)\,,$$

where (,) stands for the usual scalar product in $L^2(R^n)$.

By making use of this proposition, we shall prove the following theorem.

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0) It is convenient to write the Schrödinger operator as $-\Delta-p(x)$ instead of as the usual notation $-\Delta+p(x)$ since we are mainly concerned with an eigenvalue problem of the following form: $-\Delta u+u=\lambda p(x)u$, $p(x)>0$.