J. Math. Soc. Japan
Vol. 29, No. 1, 1977

Mean ergodic theorems for semigroups of positive linear operators

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(Received Feb. 5, 1976)

Introduction.

The mean ergodic theorem was given first by von Neumann [10], and has been generalized to semigroups of operators more general than the discrete semigroup $\{T^n : n \geqq 0\}$ by Alaoglu and Birkhoff [1], Eberlein [2], and many others, see [6, VIII. 10]. Let $\mathfrak{S}$ be a semigroup of bounded linear operators on a Banach space. Then the mean ergodic theorem is concerned with the existence and uniqueness of a fixed point of $\mathfrak{S}$ in the closed convex hull of the orbit under $\mathfrak{S}$. The main technique in the mean ergodic theorem is based on various weak compactness properties of orbits, and the (left) amenability condition for semigroups is useful in view of Day's fixed point theorem [4].

In this paper, let $(X, \mathcal{F}, m)$ be a σ-finite measure space. We shall study mean ergodic properties of semigroups of bounded linear operators on $L_1(X) = L_1(X, \mathcal{F}, m)$, and determine the structure of those semigroups for which the mean ergodic theorem holds. In particular, we shall consider amenable semigroups of uniformly bounded positive linear operators on $L_1(X)$, and also consider general semigroups of positive linear contractions on $L_1(X)$. In § 1 we shall obtain three decomposition theorems from the viewpoint of the mean ergodic theory, applying Takahashi [11, 12] and Nagel [9]. In § 2 several criteria will be given, in connection with the decompositions in § 1, which are equivalent to the condition that the mean ergodic theorem holds on the whole space $L_1(X)$. In § 3 other necessary and sufficient conditions will be given for the k-parameter semigroup and the discrete semigroup, by means of weak compactness properties of orbits.

The authors would like to express their hearty thanks to Professor H. Umegaki for his valuable suggestions.

§ 1. Decomposition theorems.

Throughout this paper, let $(X, \mathcal{F}, m)$ be a σ-finite measure space and let $L_1(X) = L_1(X, \mathcal{F}, m)$ and $L_\infty(X) = L_\infty(X, \mathcal{F}, m)$ be the usual Banach spaces of