J. Math. Soc. Japan
Vol. 28, No. 4, 1976

On finite multiplicative subgroups of simple algebras of degree 2

By Michitaka HIKARI

(Received Dec. 6, 1976)

Amitsur ([**1**]) determined all finite multiplicative subgroups of division algebras. We try to determine, more generally, all finite multiplicative subgroups of simple algebras of fixed degree. In [5] we characterized p-groups contained in the full matrix algebras $M_n(\Delta)$ of fixed degree n, where Δ is a division algebra of characteristic 0. In this paper we will study multiplicative subgroups of $M_2(\Delta)$.

In § 2 we will determine all finite nilpotent subgroups of $M_2(\Delta)$, and in § 3 all finite subgroups of $M_2(\Delta)$ with abelian Sylow 2-groups. Finally, in § 4, we will give some additional remarks.

§ 1. Preliminaries.

All division algebras considered in this paper are of characteristic 0. As usual $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ denote respectively the rational number field, the real number field, the complex number field, and the quaternion algebra over $\boldsymbol{R}$.

Let Δ be a division algebra. We denote by $M_n(\Delta)$ the full matrix algebra of degree n over Δ. By a subgroup of $M_n(\Delta)$ we mean a finite multiplicative subgroup of $M_n(\Delta)$. Further let K be a field contained in the center of Δ and let G be a subgroup of $M_n(\Delta)$. We define $V_K(G)=\{\sum \alpha_i g_i \mid \alpha_i \in K,\ g_i \in G\}$. Then $V_K(G)$ is a K-subalgebra of $M_n(\Delta)$ and there is a natural epimorphism $KG \rightarrow V_K(G)$. Hence $V_K(G)$ is a semi-simple K-subalgebra of $M_n(\Delta)$.

Let m, r be relatively prime integers, and put $s=(r-1, m)$, $t=m/s$; $n=$the minimal positive integer satisfying $r^n \equiv 1 \bmod m$. Denote by $G_{m,r}$ the group generated by two elements a, b with the relations; $a^m=1$, $b^n=a^t$ and $bab^{-1}=a^r$. Let ζ_m be a fixed primitive m-th root of unity and let $\sigma=\sigma_r$ be the automorphism of $\boldsymbol{Q}(\zeta_m)$ determined by the mapping $\zeta_m \rightarrow \zeta_m{}^r$. Let $\{\alpha_{\sigma i,\sigma j}\}$ be the factor set of $\langle\sigma\rangle$ in $\boldsymbol{Q}(\zeta_m)$ defined by

$$\alpha_{\sigma i,\sigma j}=\begin{cases} 1 & \text{when} \quad i+j<n \\ \zeta_s=\zeta_m{}^t & \text{when} \quad i+j \geqq n\,, \end{cases}$$

and denote by $A_{m,r}$ the crossed product of $\boldsymbol{Q}(\zeta_m)$ and $\langle\sigma\rangle$ by $\{\alpha_{\sigma i,\sigma j}\}$.