J. Math. Soc. Japan
Vol. 28, No. 3, 1976

Note on the positive definite integral quadratic lattice

By Michio OZEKI

(Received July 4, 1973)
(Revised Jan. 23, 1976)

§ 0. Introduction.

By a quadratic lattice L we understand a finitely generated module over $\boldsymbol{Z}$, the ring of rational integers in which a metric is given in the sense of M. Eichler [2]. The bilinear form associated to the metric is denoted by (x, y) where x and y are elements of L. If for any $x \neq 0$ in L we have $(x, x) > 0$, we shall say L is positive definite, and if it holds that $(x, y) \in \boldsymbol{Z}$ for any pair x and y in L, we shall say L is integral. Since we shall confine ourselves to the positive definite integral quadratic lattice only, we shall call such a lattice merely a lattice. Since for any element x of a lattice L (x, x) is a positive rational integer, we shall say x is m-vector when (x, x) is equal to a positive rational integer m. It is known that the sublattice generated by 2-vectors in a lattice plays an important role in the classification theory of positive definite integral quadratic lattices (E. Witt [8], M. Kneser [4], H.-V. Niemeier [5]).

The first purpose of this paper is to show that when n is an integer not smaller than 17 among all lattices of fixed rank n a lattice has the largest number of 2-vectors if and only if L contains D_n or L is equal to B_n (D_n and B_n are defined in § 1). Roughly speaking, the set of 2-vectors and 1-vectors in a lattice exibits the order of the subgroup generated by reflections in the group of units of that lattice.

Our second purpose in this paper is to prove Theorem 2 which says that if the determinant of a lattice L exceeds 2^n, where n equals to the rank of L, then the rank of the sublattice of L generated by 2-vectors is smaller than n.

Though a fair part of the results in § 2 is not new, we think it is not worthless to expose it with abbreviated proofs because some of the standpoints is not found in previous literatures as far as we know.

§ 1. Some basic notations and definitions.

We shall use $e_1, e_2, \cdots, e_m, f_1, f_2, \cdots, f_k$ or $g_1, \cdots, g_p$ as orthonormal vectors in an Euclidean space $\boldsymbol{R}^n$ of sufficiently large dimension n $(n=1, 2, 3, \cdots)$. We