J. Math. Soc. Japan
Vol. 28, No. 2, 1976

A note on automorphisms of real semisimple Lie algebras

By Takeshi HIRAI

(Received Jan. 24, 1975)

Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{g}_c$ and $\mathfrak{h}_c$ the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and σ the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$. Denote by $W(\mathfrak{h}_c)$ the Weyl group of $\mathfrak{g}_c$ acting on $\mathfrak{h}_c$, and let $W_\sigma(\mathfrak{h})$ be the subgroup of $W(\mathfrak{h}_c)$ consisting of elements leaving $\mathfrak{h}$ stable or

$$W_\sigma(\mathfrak{h})=\{w\in W(\mathfrak{h}_c)\,;\ w\circ\sigma=\sigma\circ w \quad \text{on } \mathfrak{h}_c\}\ .$$

The purpose of this paper is to study the structure of the group $W_\sigma(\mathfrak{h})$, as was done in [7, Appendix] for such $\mathfrak{h}$ which has the maximal vector part.

§ 1. Statement of Theorem.

For any linear form λ on $\mathfrak{h}_c$, we define $\sigma\lambda$ as $(\sigma\lambda)(X)=\overline{\lambda(\sigma X)}$ $(X\in\mathfrak{h}_c)$, where $\bar{a}$ denotes the conjugate number of $a\in \boldsymbol{C}$. If α is a root of $(\mathfrak{g}_c, \mathfrak{h}_c)$, then so is $\sigma\alpha$. Let $\mathfrak{r}$ be the root system of $(\mathfrak{g}_c, \mathfrak{h}_c)$, then it is a σ-system of roots in the sense of [1] with the involutive automorphism σ. A root α is called real or imaginary if $\sigma\alpha=\alpha$ or $\sigma\alpha=-\alpha$ respectively. We see that α is real or imaginary if and only if it takes only real or purely imaginary values on $\mathfrak{h}$ respectively. Denote by $\mathfrak{r}_R$ and $\mathfrak{r}_I$ the sets of all real or imaginary roots in $\mathfrak{r}$ respectively. Let $W_R(\mathfrak{h})$ and $W_I(\mathfrak{h})$ be the groups generated by $\{s_\alpha\,;\alpha\in\mathfrak{r}_R\}$ and $\{s_\alpha\,;\alpha\in\mathfrak{r}_I\}$ respectively, where s_α denotes the reflexion corresponding to a root α. Then they are normal subgroups of $W_\sigma(\mathfrak{h})$, because $W_\sigma(\mathfrak{h})$ leaves $\mathfrak{r}_R$ and $\mathfrak{r}_I$ stable.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. For a subset F of $\mathfrak{g}$, let $N_G(F)$ and $Z_G(F)$ be the normalizer and the centralizer of F in G respectively, and put $W_G(F)=N_G(F)/Z_G(F)$. Then $W_G(\mathfrak{h})$ is considered canonically as a subgroup of $W(\mathfrak{h}_c)$, and also of $W_\sigma(\mathfrak{h})$ since $\mathrm{Ad}\,(g)\circ\sigma=\sigma\circ\mathrm{Ad}\,(g)$ on $\mathfrak{g}_c$ for any $g\in G$. We know that $W_R(\mathfrak{h})$ is a subgroup of $W_G(\mathfrak{h})$ (see for instance [2, p. 256]). An imaginary root α is called compact if $(\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}+[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}])\cap\mathfrak{g}$ is isomorphic to $\mathfrak{su}(2)$, where $\mathfrak{g}_{\pm\alpha}$ are the spaces of root vectors corresponding to $\pm\alpha$. If $\alpha\in\mathfrak{r}_I$ is compact, then $s_\alpha\in W_G(\mathfrak{h})$ [2, p. 256]. We define the vector part $\mathfrak{h}^-$ and the toroidal part $\mathfrak{h}^+$ of $\mathfrak{h}$ as follows: