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Nonlinear variational inequalities and fixed point theorems

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§ 1. Introduction.

It was proved by Hartman and Stampacchia [8] in 1966 that if $T: R^n \rightarrow R^n$ is a continuous mapping on a compact, convex subset X of R^n, then there exists $x_0 \in X$ such that $\langle Tx_0, x_0 - x \rangle \geqq 0$ for all $x \in X$. This remarkable result has been investigated and generalized in various points of views by Browder [**1**], [**2**], Moré [**10**] and others. For example, Browder extended this theorem to the case of which our considering mappings T are of a compact convex subset X of a topological vector space E into the dual space E^*; see Theorem 2 of [**2**]. In § 2 of this paper, we shall obtain two generalizations of this Browder's theorem. One of them is Lemma 1 that has various applications. The other is Theorem 3 that generalizes the Browder's result to closed and convex sets in topological vector spaces. We shall also make use of Theorem 3 to prove Theorem 4 that generalizes Moré's theorem [**10**, Theorem 2.4]. In § 3, using Lemma 1, we shall prove some fixed point theorems. Theorem 5 and Theorem 9 extend Browder's fixed point theorems [**1**, Theorem 1], [**2**, Theorem 3]. In § 4, we shall discuss Sion's minimax theorem and Terkelsen's minimax theorem. At first, we shall show that Sion's theorem follows simply from the fundamental and useful theorem of Browder [**2**, Theorem 1]. Furthermore, we state a necessary and sufficient condition that a minimax condition holds. Using this, we shall generalize Terkelsen's minimax theorem; see Theorems 16 and 17. In § 5, we give another proof for Fan's theorem [**5**] concerning systems of convex inequalities. The proof is simple. Furthermore, using this Fan's result, we prove Fan's minimax theorem [**4**] and also obtain a generalization of the result of Browder [**2**, Lemma 1]; see Theorems 18, 19 and 20. At last, by Lemma 1, we generalize Browder's theorem [**2**, Theorem 6] for multi valued mappings; see Theorem 21. By the same methods, we shall also generalize Kakutani's fixed point theorem [**9**]; see Theorem 22.

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