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On some improvements of the Brun-Titchmarsh theorem, III

Dedicated to Professor T. Tatuzawa on his 60th birthday

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§1. Introduction.

In our preceding papers [5], [6] we have established among other things that, denoting as usual by $\pi(x;q,a)$ the number of primes less than x and congruent to a (mod q), we have the inequality

$$\pi(x;q,a)\leqq\frac{2x}{\varphi(q)\log\frac{x}{\sqrt{q}}}\Big(1+O\Big(\frac{\log\log x}{\log x}\Big)\Big), \tag{1}$$

for all a (mod q) and for almost all a (mod q) when $q\leqq x^{2/5}$ and $q\leqq x^{1-\varepsilon}$, respectively. The former case is the first substantial improvement of the Brun-Titchmarsh theorem and also of the recent result of Montgomery and others [4]. The later case is an improvement of a result of Hooley [2].

Roughly speaking, these are concerning the fixed modulus q and moving residue a. And it may be interesting to consider the dual problem in which the residue a is fixed and the modulus q runs over a certain interval. Then we may expect that the Brun-Titchmarsh theorem can be improved for almost all q. The first result in this field has been obtained in the above quoted paper of Hooley. He has proved that, if a is a fixed non-zero integer, K any positive constant and $W\leqq q<2W$, $(q,a)=1$, then we have

$$\pi(x;q,a)\leqq\begin{cases}\dfrac{(1+\varepsilon)x}{\varphi(q)\log\Big\{\Big(\dfrac{x^2}{W}\Big)^{1/6}\Big\}} & \text{for}\quad x^{1/2}\leqq W\leqq x^{4/5}\\[2ex] \dfrac{(1+\varepsilon)x}{\varphi(q)\log\dfrac{x}{W}} & \text{for}\quad x^{4/5}\leqq W\leqq x^{1-\varepsilon},\end{cases} \tag{2}$$

save for at most $W(\log x)^{-K}$ exceptional values of q.

This problem has certain similarity to the celebrated mean-value prime number theorem of Bombieri [1] (see also A. I. Vinogradov [8]), and the result of Hooley has definite interest, since Bombieri's theorem and even the extended