J. Math. Soc. Japan
Vol. 27, No. 1, 1975

On a class number relation of imaginary abelian fields

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(Received March 5, 1974)

§ 1. Introduction.

Let k_0 be the cyclotomic field $Q(\zeta_p)$ generated by a primitive p-th root of unity ζ_p over the rationals Q, where p is a prime number >3. Let k_0^+ be the maximal real subfield of k_0. Recently, Metsänkylä [7], [8] gave a relation between the class number h_0^+ of k_0^+ and the relative class number h_0^- of k_0/k_0^+ in the form

$$h_0^- \equiv G h_0^+ \pmod{p}, \tag{1}$$

where G is an explicitly given integer.

In this paper we shall generalize this relation (1) to the class number factors h_K^- and h_K^+ of certain imaginary abelian number field K over Q (Theorems 1, 2, § 3), by means of continuity of p-adic L-functions [4], [5] and the p-adic formulas for h_K^+ [6] and h_K^-. For this purpose, we use some results connected with p-adic L-functions which are derived by Fresnel [2] and simplified by Shiratani [10].

Denote by q a square-free integer >1 and by $d=3q$ the discriminant of a real quadratic number field. Consider the real field $Q(\sqrt{3q})$ and the imaginary field $Q(\sqrt{-q})$. As an application of our Theorems 1, 2, we shall obtain a classical result ((21), § 4) of Ankeny-Artin-Chowla [1], which states a congruence relation modulo 3 between the class numbers of $Q(\sqrt{3q})$ and $Q(\sqrt{-q})$ for $q \equiv 1 \pmod{3}$. Furthermore in § 4 we shall give some similar results other than (21).

§ 2. Relations between $L_p(0, \chi)$ and $L_p(1, \chi)$.

Let p be an arbitrarily fixed prime number, Q_p the field of rational p-adic numbers and Z_p the ring of rational p-adic integers. Let χ be an even Dirichlet character and $L_p(s, \chi)$ the p-adic L-function for χ. The function $L_p(s, \chi)$ is a continuous function of $s \in Z_p$ $(s \neq 1)$, and if χ is not the principal character, then $L_p(s, \chi)$ is continuous at $s=1$ [4], [5]. A Dirichlet character