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A construction of β-normal sequences

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In this paper we define the normality of sequences in the scale of not necessarily integral β and give a construction of β-normal sequences as a generalization of Champernowne's construction of normal sequences.

Let $\beta>1$ be a fixed real number. Define a transformation T_β on the unit interval, which we call β-transformation, as follows: $T_\beta x=\beta x-[\beta x]$, $0\leqq x<1$, where $[z]$ is the integral part of z. Then T_β has an invariant probability measure μ_β, under which T_β is ergodic, such that

$$1-\beta^{-1}<\frac{d\mu_\beta}{dx}=\frac{1}{E_\beta}\sum_{n=0}^{\infty}\frac{c_n(x)}{\beta^n}<(1-\beta^{-1})^{-1},$$

where

$$c_n(x)=\begin{cases}1 & \text{if} \quad x<T^n 1,\\ 0 & \text{if} \quad x\geqq T^n 1,\end{cases}$$

$$T^0 1=1, \qquad T^n 1=T_\beta^{n-1}(\beta-[\beta]),$$

and E_β is the normalizing constant (see [2]). Recently the first named author and Y. Takahashi investigated in [1] the β-transformations as a class of symbolic dynamics and obtained various new results. Our theorem (in this paper) is a byproduct of these results.

Consider the β-adic expansion of a real number x, $0\leqq x<1$, i.e.

$$x=\sum_{n=0}^{\infty}\omega_n(x)\beta^{-n-1}$$

where $\omega_n(x)=[\beta T^n x]$, $n\geqq 0$. Then through the mapping $\pi_\beta(x)=\omega_0(x)\omega_1(x)\cdots$ β-transformation is isomorphic to a shift on the one-sided product space A^N where A is the state space $\{0, 1, \cdots, \beta_0\}$ and β_0 is the greatest integer less than β. Of course the measure on A^N is generated by $\pi_\beta\pi_\beta^{-1}$, which we again denote by μ_β. Now we define the β-normality of a sequence in A^N.

A sequence $b=b_0b_1b_2\cdots$ in A^N is said to be β-normal if for any positive integer k and any word $u=u_1u_2\cdots u_k$ of length k we have

$$\lim_{n\to\infty} n^{-1}F_n(u)=\mu_\beta(u)$$