J. Math. Soc. Japan
Vol. 26, No. 3, 1974

Mixed-type boundary conditions for second order elliptic differential equations

By Yoshio KATO

(Received March 5, 1973)

§0. Introduction.

Let Ω be a bounded domain in R^m with C^∞ boundary Γ of dimension $m-1$ and let there be given two differential operators A on $\bar{\Omega}=\Omega\cup\Gamma$,

$$A=\sum_{|\mu|,\,|\nu|\leqq 1} D^\mu a_{\nu\mu}(x)D^\nu\,, \tag{0.1}$$

and B_1 on Γ,

$$B_1=a_0(x)\frac{\partial}{\partial n}+b(x, D)\,, \tag{0.2}$$

where n denotes the exterior normal of Γ and $b(x, D)$ is a tangential differential operator of first order on Γ. The notations are the usual ones: $\nu=(\nu_1, \cdots, \nu_m)$ with non-negative integers ν_j, $|\nu|=\nu_1+\cdots+\nu_m$, $D=(D_1, \cdots, D_m)$ with $D_j=-i\partial/\partial x_j$ and $D^\nu=D_1^{\nu_1}\cdots D_m^{\nu_m}$. All coefficients are assumed to be complex-valued and C^∞ in their sets of definition indicated. It is also assumed that A is elliptic and moreover that $a_0(x)\neq 0$ for all $x\in\Gamma$.

Many authors had studied the mixed problem: To find u such that

$$\begin{cases} Au=f & \text{in } \Omega \\ B_1u=0 & \text{on } \Gamma_1 \\ u=0 & \text{on } \Gamma_2 \end{cases} \tag{0.3}$$

when f is given, where Γ_1 and Γ_2 are two open sets of Γ such that $\bar{\Gamma}_1\cup\bar{\Gamma}_2=\Gamma$ and $\Gamma_1\cup\Gamma_2=\emptyset$. But it seems to be more natural to replace and generalize (0.3) with

$$\begin{cases} Au=f & \text{in } \Omega \\ \alpha B_1u+\beta u=0 & \text{on } \Gamma\,, \end{cases} \tag{0.4}$$

where α and β are two functions on Γ such that $\alpha\geqq 0$, $\beta\geqq 0$ and $\alpha+\beta=1$ on Γ. The first reason is that for the problem (0.3) no regularity can hold, i.e. even if f is smooth, u may not be smooth (this fact is pointed out in [**1**, **5**]), while the regularity holds for the problem (0.4), provided α is smooth. The second is that all solutions of (0.3) are approximated by solutions u_n of