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On some improvements of the Brun-Titchmarsh theorem

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§ 1. Introduction.

The so-called Brun-Titchmarsh theorem states that there is an absolute constant C_0 such that

$$\pi(x; q, l) \leqq C_0 \frac{x}{\varphi(q) \log \frac{x}{q}}, \tag{1.1}$$

where $\pi(x; q, l)$ is defined as usual to be the number of primes not exceeding x that are congruent to $l \bmod q$. This estimation holds uniformly for all $q < x$ with x/q sufficiently large and for all $l \bmod q$ with $(q, l) = 1$.

The most important feature of this theorem is that it holds for a quite wide range of q and in this respect it surpasses any results obtained by analytic methods. Actually this inequality may be the strongest tool to attack various problems in the theory of numbers, apart from the mean value theorem of Bombieri.

Although the asymptotic formula of $\pi(x; q, l)$, which holds uniformly for smaller q, can be obtained by analytic methods, the result (also its aesthetic value) is mared by the exceptional zero of Dirichlet's L-functions. The elimination of this zero remains still one of the deepest problems in the theory of numbers.

In this respect the reduction of the value of the constant C_0 of (1.1) has a very significant meaning, for if we could show that

$$C_0 \leqq 2 - \eta \tag{1.2}$$

with an effectively calculable positive η for at least $\frac{\varphi(q)}{2} + 1$ reduced residue classes $l \bmod q$ under the condition

$$q \leqq \exp\left(c_1 \frac{\log x}{\log \log x}\right)$$

then we would be able to prove the extremely valuable inequality

$$\beta \leqq 1 - \frac{c_2}{\log (q+3) \log \log (q+3)}$$