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# An extended relativization theorem

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Let $L$ be the first order infinitary logic $L_{\omega_1\omega}$ (cf. Feferman [2]) without individual constants or function symbols, $V$ an unary predicate symbol not in $L$, and $L(V)$ the first order logic obtained from $L$ by adding $V$ as a new predicate symbol. For any formula $A$ in $L$, we denote by $A^V$, the formula in $L(V)$ obtained from $A$ by relativizing every quantifier in $A$ by $V$. Then, we have

RELATIVIZATION THEOREM. *Let $A$, $B$ be any sentences in $L$. If* $(\exists v)V(v) \underset{L(V)}{\vdash} A^V \supset B$, *then there is an existential sentence $C$ such that* $\underset{L}{\vdash} A \supset C$ *and* $\underset{L}{\vdash} C \supset B$. (Cf. Motohashi [5].)

This theorem is a syntactical counterpart of Łos-Tarski's theorem on extensions in model theory.

In this paper, we shall extend this theorem to a form which can be considered as a syntactical counterpart of Chang-Łos' theorem on $\omega$-ascending unions (cf. Chang [1]), Keisler's theorem on unions (cf. Keisler [3]) and Nebres' theorem on unions (cf. Nebres [8]). To accomplish this purpose we use a binary predicate symbol $U(*, *)$ instead of the unary predicate symbol $V(*)$ and a two sorted first order logic $L(U)$, which has two kinds of (individual) variables such as $x, y, \cdots$ (free variables of type 1), $u, v, \cdots$ (bound variables of type 1), $\alpha$, $\beta$, $\cdots$ (free variables of type 2) and $\xi$, $\eta$, $\cdots$ (bound variables of type 2). We assume that every variables in $L$ is of type 1 and every atomic formula in $L(U)$ has one of the forms: $U(\alpha, x)$ or $P(x_1, \cdots, x_n)$, where $P$ is a predicate symbol in $L$. For any free variable $\alpha$ of type 2 and any formula $A$ in $L$, $A^\alpha$ is the formula in $L(U)$ obtained from $A$ by relativizing every quantifier in $A$ by $U(\alpha, *)$. Then we have

MAIN THEOREM. *Let $k$ be a non negative integer and $A$, $B$ sentences in $L$. If* $(\forall\xi)(\exists u)U(\xi, u), (\forall u_1)\cdots(\forall u_k)(\exists\xi)(U(\xi, u_1)\wedge\cdots\wedge U(\xi, u_k)) \underset{L(U)}{\vdash} (\forall\xi)A^\xi \supset B$, *then there is a sentence $C$ of the form $C=(\forall u_1)\cdots(\forall u_k)E(u_1, \cdots, u_k)$ such that* $\underset{L}{\vdash} A\supset C$ *and* $\underset{L}{\vdash} C\supset B$, *where $E(x_1, \cdots, x_k)$ is an existential formula in $L$.*

This theorem implies the relativization theorem as a special case $k=0$. For simplicity we shall prove this theorem in the case $k=1$, $L$ is finitary and has no equality symbol: There is no difficulty to prove it in general