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Model theory on a positive second order logic with countable conjunctions and disjunctions

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Introduction.

This paper is a sequel to our paper [8], in Chapter V of which we developed a general theory of so-called "preservation theorem" without using any model theoretic notions, so that we can apply it to different kinds of logics. In this paper, we shall show some applications of it to the model theory on a positive second order logic $\mathfrak{L}$ (in the sense of [17]) with countable conjunctions and disjunctions. (Cf. Theorem 4.1 and Theorem 4.2 in §4.)

Suppose $(\exists\xi)\varphi(\xi)$, $(\forall\eta)\psi(\eta)$, φ_1, ψ_1 are sentences in the second order logic such that $\varphi(\xi)$, $\psi(\eta)$, φ_1, ψ_1 have no second order quantifiers and ξ, η are second order variables. Notice that the sentence $(\exists\xi)\varphi(\xi)\supset(\forall\eta)\psi(\eta)$ is a sentence in the positive second order logic $\mathfrak{L}$. Hence, Craig's interpolation theorem can be expressed in the following form:

(1) If $\vdash(\exists\xi)\varphi(\xi)\supset(\forall\eta)\psi(\eta)$, then $\vdash(\exists\xi)\varphi(\xi)\supset\theta$ and $\vdash\theta\supset(\forall\eta)\psi(\eta)$ for some *first order* sentence θ.

Also, Łos-Tarski's theorem on extension can be expressed in the following form:

(2) If every extension of a model of φ_1 is a model of ψ_1 then $\vdash\varphi_1\supset\theta$ and $\vdash\theta\supset\psi_1$ for some *existential* sentence θ.

Combining (1) and (2), we can get

(3) If every extension of a model of $(\exists\xi)\varphi(\xi)$ is a model of $(\forall\eta)\psi(\eta)$, then $\vdash(\exists\xi)\varphi(\xi)\supset\theta$ and $\vdash\theta\supset(\forall\eta)\psi(\eta)$ for some *first order existential* sentence θ.

This is an example of preservation theorems in the positive second order logic $\mathfrak{L}$. Our Theorem 4.1 is a generalization of the preservation theorems of the form (3) to $L_{\omega_1\omega}$.

On the other hand, Tarski's theorem on PC_δ-class can be expressed in the following form:

(4) The class of substructures of models of $(\exists\xi)\varphi(\xi)$ is an EC_δ-class.

Our Theorem 4.2 is a generalization of infinitary analogues of (4) to $L_{\omega_1\omega}$. After some preparations in §1, 2 and 3, we shall prove these theorems in