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A topological invariant of substitution minimal sets

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§ 0. Introduction.

In this paper, we examine the ergodic properties of a bisequence over some finite set of symbols which is generated by a *substitution*. By a substitution, we mean a mapping which maps each symbol to a sequence of some common length ($\geqq 2$) of the symbols. For example, consider a substitution $\theta: 0 \to 0\ 1,\ 1 \to 1\ 0$ which is defined on $\{0, 1\}$. Define substitutions $\theta^2, \theta^3, \cdots$, as follows:

$$\theta^2(0) = \theta(0\ 1) = \theta(0)\theta(1) = 0\ 1\ 1\ 0\,,$$
$$\theta^2(1) = \theta(1\ 0) = \theta(1)\theta(0) = 1\ 0\ 0\ 1\,,$$
$$\theta^3(0) = \theta(0\ 1\ 1\ 0) = \theta(0)\theta(1)\theta(1)\theta(0)$$
$$= 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\,,$$
$$\vdots$$

A bisequence $\alpha = \cdots 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1 * 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1 \cdots$ which is known as the Morse sequence is defined as the limit of extensions:

$$1 * 0 \prec \theta^2(1 * 0) \equiv \theta^2(1) * \theta^2(0) = 1\ 0\ 0\ 1 * 0\ 1\ 1\ 0$$
$$\prec \theta^4(1 * 0) \prec \theta^6(1 * 0) \prec \cdots,$$

where "$*$" denotes the "center" of bisequences. In this case, it holds that $\theta^2(\alpha) = \alpha$. Generally speaking, given any substitution θ over some finite set D, a bisequence α over D is said to be *generated* by θ if $\theta^k(\alpha) = \alpha$ for some integer $k \geqq 1$. It is known ([3]) that for any substitution θ, there exists at least one almost periodic sequence generated by θ. Moreover, it can be proved (from Lemma 1$\sim$3) that if θ satisfies Condition $\sharp$ defined in Section 1, all almost periodic sequences generated by θ belong to a common minimal set of the shift dynamical system over D. Such a minimal set S as above is unique and characterized as a minimal set S for which $\theta(S) \subset S$ holds. In this case, denote $S = W(\theta)$. Let Θ be a set of all substitutions defined on $\{0, 1, \cdots, r-1\}$ for some integer $r \geqq 1$ which satisfy Condition $\sharp$. We introduce a computable (in the sense of the recursive function theory) function B called