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On Bazilevič functions of bounded boundary rotation

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§1. Introduction.

Let

(1) $$f(z)=\left\{\frac{\beta}{1+\alpha^2}\int_0^z(h(\zeta)-\alpha i)\zeta^{[-\alpha\beta i/(1+\alpha^2)]-1}g(\zeta)^{\beta/(1+\alpha^2)}d\zeta\right\}^{(1+\alpha i)/\beta}$$

where $h(z)=1+\sum_{n=1}^{\infty}c_nz^n$ satisfies $\mathrm{Re}\,h(z)>0$ in $|z|<1$, $g(z)$ is starlike in $|z|<1$, α is any real number and $\beta>0$.

Bazilevič [1] introduced the above class of functions and showed that each such function is univalent in $|z|<1$.

Let $\alpha=0$ in (1). On differentiating we get

(2) $$zf'(z)=f(z)^{1-\beta}g(z)^{\beta}h(z)$$

and

(3) $$\mathrm{Re}\,h(z)=\mathrm{Re}\,(zf'(z)/f(z)^{1-\beta}g(z)^{\beta})>0 \quad \text{in} \quad |z|<1.$$

Thomas [6] called a function satisfying the condition (3) a Bazilevič function of type β. Let $C(r)$ denote the curve which is the image of the circle $|z|=r<1$ under the mapping $w=f(z)$, $L(r)$ the length of $C(r)$ and $A(r)$ the area enclosed by the curve $C(r)$. Let $M(r)=\max_{|z|=r}|f(z)|$.

Hayman [2] gave an example of a bounded starlike function satisfying

$$\lim_{r\to1}\sup\frac{L(r)}{\log 1/(1-r)}>0.$$

In [7] Thomas gave the following open problems: Does there exist a starlike function for which

$$\lim_{r\to1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{L(r)}{M(r)\log 1/(1-r)}>0$$

or

$$\lim_{r\to1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{L(r)}{\sqrt{A(r)}\log 1/(1-r)}>0.$$

In this paper the author gives some results concerning this and others.