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Some potential theory of processes with stationary independent increments by means of the Schwartz distribution theory

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§ 0. Introduction.

In this paper we will be primarily concerned with the most general processes with stationary independent increments on the real line $\boldsymbol{R}$. All the results are valid for the higher dimensional such processes without change. It is only for saving the notation that we restrict ourselves to the one-dimensional processes.

We will summarize the contents of the paper with the main results being picked up in (A) to (D).

§ 1 through § 3 are of quite analytic character. Let $(\mu_t)_{t\geqq 0}$ be a convolution semi-group of probability measures on $\boldsymbol{R}$. Let $(P_t)_{t\geqq 0}$ be the semi-group of Markov kernels defined by $P_t f=\int f(x+y)\mu_t(dy)$ and $(U_\lambda)_{\lambda>0}$, the resolvent of (P_t). C_0 stands for the space of continuous functions vanishing at infinity and $(\mathcal{D}'_{L^p})$, $1\leqq p\leqq\infty$, the spaces of distributions introduced by L. Schwartz [10; Chap. VI, § 8]. It has been known that, for every $f\in C_0^2=\{f\in C_0;\ f', f''\in C_0\}$, the uniform limit of $t^{-1}[P_t f-f]$ as $t\to 0$ is given by

$$(0.1)\qquad Af(x):=af'(x)+\frac{\sigma^2}{2}f''(x)+\int_{\boldsymbol{R}\setminus\{0\}}\Big[f(x+y)-f(x)-\frac{y}{1+y^2}f'(x)\Big]\nu(dy),$$

where $a\in\boldsymbol{R}$, $\sigma^2\geqq 0$ and ν is the so-called Lévy measure. L. Schwartz's basic results on the spaces $(\mathcal{D}'_{L^p})$ make it possible to extend the operators P_t, U_λ and A to those on the space of "bounded distributions" $(\mathcal{D}'_{L^\infty})$. Hence, for example, if f is a bounded function, Af is well-defined and belongs to $(\mathcal{D}'_{L^\infty})$. We see that Af is of the form $\tilde{A}^0 * f$ by means of an element $\tilde{A}^0\in(\mathcal{D}'_{L^1})$, where "$*$" means the convolution. It should be noted that C. S. Herz [5] called this distribution $\tilde{A}^0$ a *generalized Laplacian* and studied its structure in a little different context from ours.

The following theorem is fundamental throughout the paper and it is proved in § 2.

(A) For every $\lambda>0$ and $f\in(\mathcal{D}'_{L^\infty})$, $u=U_\lambda f$ is the unique solution in $(\mathcal{D}'_{L^\infty})$