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A formula on some odd-dimensional Riemannian manifolds related to the Gauss-Bonnet formula

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§ 1. Introduction.

Let (N^{2n}, g) be a compact orientable Riemannian manifold of $2n$-dimension. The generalised Gauss-Bonnet formula is

$$\frac{(-1)^n}{2^{2n}\pi^n n!}\int_N \sum \varepsilon_{i_1\cdots i_{2n}} \Omega'_{i_1 i_2} \wedge \cdots \wedge \Omega'_{i_{2n-1} i_{2n}} = \chi(N), \tag{1.1}$$

where Ω'_{ij} denote the curvature forms and $\chi(N)$ is the Euler-Poincaré characteristic. The left hand side of (1.1) is a differential geometric or Riemannian geometric quantity and the right hand side is a topological quantity. In (1.1), even dimensionality is essential.

For a compact orientable Riemannian manifold (M^{2n+1}, g) of odd dimension, we have $\chi(M)=0$. This shows that $M=M^{2n+1}$ admits a vector field ξ with no singular points. If we try to find some formula on (M^{2n+1}, g) analogous to (1.1), some restriction on this ξ may be necessary and it might be hoped that the right hand side is a linear combination of Betti numbers.

We assume that $\xi=e_0$ is a unit vector field. Let w_0 be the 1-form dual to e_0 with respect to g. Then we have local fields of orthonormal vectors $e_0, e_1, \cdots, e_{2n}$ and the dual $w_0, w_1, \cdots, w_{2n}$. We call this frame field a ξ-frame field. By Ω_{AB} $(A, B=0, 1, \cdots, 2n)$ we denote the curvature forms with respect to the above frame field. By $\beta_r(M)$ we denote the r-th Betti number of M. In this paper we have

THEOREM A. *Let (M^{2n+1}, g) be a compact Riemannian manifold admitting a unit Killing vector ξ and let (e_0, e_i) be a ξ-frame field. Assume that*

$$\Omega_{0i} = w_i \wedge w_0, \qquad i=1, \cdots, 2n, \tag{1.2}$$

and that each trajectory of ξ is of constant length $l(\xi)$. Then

$$\frac{(-1)^n}{l(\xi)2^{2n}\pi^n n!}\int_M F(\Omega_{ij}, w_0) = \sum_{r=0}^{n}(n+1-r)(-1)^r \beta_r(M), \tag{1.3}$$

where, putting $dw_0 = \sum \varphi_{AB} w_A \wedge w_B$,

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