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Localization theorem for holomorphic approximation on open Riemann surfaces

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§ 1. Introduction.

Let R be an open Riemann surface and K a compact subset of R. Let $C(K)$ be the class of complex valued continuous functions on K. A function f of $C(K)$ is said to be in $H(K)$, if f is the uniform limit on K of functions, each holomorphic in some neighborhood of K.

The localization theorem is the following

THEOREM A. *Let f be a function of $C(K)$. Suppose, for every point P of K, there is a neighborhood U_P of P such that $f|_{(\bar{U}_P \cap K)} \in H(\bar{U}_P \cap K)$. Then f is in $H(K)$.*

This theorem was proved in Bishop [2] and Kodama [5]. Garnett simplified the proof in the plane case [3].

In this note, we shall give two new proofs of Theorem A. The first proof is based on the solution of $\bar{\partial}$-problem with bounded estimate. The second one is a generalization of Garnett's method. Through both proofs, the elementary differential (Behnke-Stein [1]) plays the important role. In Section 2, we shall prove a generalization of Mergelyan's theorem for rational approximation [6] to open Riemann surface. In Section 8, we shall make a remark about the higher dimensional case.

§ 2. An approximation theorem.

Let $H(K, R)$ be the class of functions on K which are uniform limits on K of functions, each holomorphic on R. Let $A(K)$ be the class of functions of $C(K)$ which are holomorphic in the interior of K. As an application of Theorem A, we have the following

THEOREM B. *Let ρ be a metric on R. Suppose there is a positive constant k such that every component of $R \backslash K$ has ρ-diameter not less than k. Then $A(K) = H(K)$. In particular, if $R \backslash K$ has no relatively compact component, then $A(K) = H(K, R)$.*

PROOF. Let P be any point of K, U_P be a coordinate neighborhood of P