J. Math. Soc. Japan
Vol. 23, No. 4, 1971

Deformations of compact complex surfaces III

By Shigeru IITAKA

(Received Dec. 12, 1970)

§ 1. Introduction.

This is a continuation of the papers "Deformations of compact complex surfaces" and "Deformations of compact complex surfaces II" to which we refer as Parts I and II, respectively. We studied topological properties of plurigenera of compact complex surfaces and obtained Theorem II in Part I and Theorem IV in Part II. Moreover, we showed the invariance of plurigenera of compact complex surfaces under deformations in Theorem III in Part II. In this paper, we first investigate the structure of compact complex manifolds of dimension 4, and we prove Theorem V which would be important in the future study of bimeromorphic classification of compact complex manifolds. Furthermore, we shall show that the plurigenera of an elliptic surface are determined by the homotopy invariants of the surface, if its fundamental group is neither a finite abelian group generated by at most two elements nor a dihedral group of order $4k$, $k \geqq 1$. In order to prove the above assertion, we determine the structure of the fundamental groups of elliptic surfaces. We recall that the fundamental group of an algebraic curve determines its genus $(=h^{1,0})$ and conversely the genus determines the abstract structure of the fundamental group. In the case of surfaces, such a connection between topology and genera[1] might be lost. However if we restrict ourselves to elliptic surfaces, we can say that the fundamental group of any elliptic surface determines its plurigenera completely with minor exceptions. Conversely, the essential structure of any elliptic surface of general type can be given by use of its plurigenera, as will be shown later.

§ 2. Statement of the results.

We employ the notation and the terminology of Parts I and II. Thus, by a (compact complex) surface we mean a connected compact complex manifold of dimension 2. In [8], S. Kawai has developed a theory of bimero-

1) Here, by genera we mean the basic discrete complex analytic invariants of compact complex manifolds.