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Orbits of one-parameter groups III (Lie group case)

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§ 1. Introduction

This paper is a continuation of the previous paper, [5] M. Goto, *Orbits of one-parameter groups* II, which will be quoted as *Orbits* II, and the main purpose of this paper is to prove the following theorem.

THEOREM. *Let $\mathcal{G}$ be a Lie group. Let $\mathcal{L}$ be an analytic subgroup, and let $\mathcal{X}$ be a one-parameter subgroup, of $\mathcal{G}$. Then either*

(a) *$\mathcal{X}$ is a closed straight line and $\mathcal{X}\bar{\mathcal{L}}$ is topologically the same as the direct product $\mathcal{X}\times\bar{\mathcal{L}}$, or*

(b) *We can give a toral group structure to the set $\overline{\mathcal{X}\mathcal{L}}/\bar{\mathcal{L}}$ such that $\mathcal{X}\bar{\mathcal{L}}/\bar{\mathcal{L}}$ becomes an everywhere dense one-parameter subgroup in it.*

The theorem was proved for the general linear group $\mathcal{GL}(n, R)$, in a slightly weaker form (Theorem 1 in *Orbits* II), and it can be applied for all analytic subgroups of $\mathcal{GL}(n, R)$[2)]. However, in order to prove the theorem for a closed analytic subgroup $\mathcal{G}$ of $\mathcal{GL}(n, R)$, we need some groups which are not in $\mathcal{G}$, but in the algebraic hull of $\mathcal{G}$.

Hence in order to extend the method in *Orbits* II to general analytic groups, it was necessary to find a suitable analytic group $\mathcal{S}$ which contains the given $\mathcal{G}$ and all the groups which appear in the process of the proof. For the purpose, we introduce the notion of *semi-algebraic* subgroups of $\mathcal{GL}(n, R)$ and *adjoint semi-algebraic* analytic groups in § 2. For a given analytic group $\mathcal{G}$, we can find an adjoint semi-algebraic group $\mathcal{S}$ which contains $\mathcal{G}$ as a closed normal subgroup by (3.4). Thus, roughly speaking, by considering the adjoint representation of $\mathcal{S}$, we can reduce the problem into the case of linear groups. The proof of the Theorem is given in § 5 and § 6.

In § 4 we shall give some lemmas, which are based on "category argument" of locally compact groups, and which make the brute force part of

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2) By a theorem in Goto [4], every analytic subgroup of $\mathcal{GL}(n, R)$ is isomorphic with a *closed* subgroup of $\mathcal{GL}(m, R)$ for a sufficiently large m.