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Radicals of gamma rings

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§ 1. Introduction

Let M and Γ be additive abelian groups. If for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following conditions are satisfied,

(1) $a\alpha b \in M$

(2) $(a+b)\alpha c = a\alpha c + b\alpha c$
$a(\alpha+\beta)b = a\alpha b + a\beta b$
$a\alpha(b+c) = a\alpha b + a\alpha c$

(3) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then, following Barnes [1], M is called a Γ-ring. If these conditions are strengthened to,

(1′) $a\alpha b \in M$, $\alpha a\beta \in \Gamma$

(2′) same as (2)

(3′) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$

(4′) $x\gamma y = 0$ for all $x, y \in M$ implies $\gamma = 0$,

then M is called a Γ-ring in the sense of Nobusawa [5].

Any ring can be regarded as a Γ-ring by suitably choosing Γ. Many fundamental results in ring theory have been extended to Γ-rings: Nobusawa [5] proved the analogues of the Wedderburn-Artin theorems for simple Γ-rings and for semi-simple Γ-rings (but see [4]); Barnes [1] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings; Luh [3, 4] gave a generalization of the Jacobson structure theorem for primitive Γ-rings having minimum one-sided ideals, and obtained several other structure theorems for simple Γ-rings.

In this paper the notions of Jacobson radical, Levitzki nil radical, nil radical and strongly nilpotent radical for Γ-rings are introduced, and Barnes' [1] prime radical is studied further. Inclusion relations for these radicals are obtained, and it is shown that the radicals all coincide in the case of a Γ-ring which satisfies the descending chain condition on one-sided ideals. The other usual radical properties from ring theory are similarly considered.

For all notions relevant to ring theory we refer to [2].