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Hecke operators in cohomology of groups

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Given a group G, with a subgroup Γ, one can always formulate the so-called Hecke rings whose elements are certain double cosets, called Hecke operators as introduced by Shimura in [4]. The study of the action of Hecke operators on the cohomology groups $H^k(\Gamma, \rho)$ with a linear representation ρ of G, defined by Kuga in [2], appears to be important in the number theory of automorphic forms, in the formulation of various "trace formulas", when the groups were Lie groups with discrete subgroups Γ, where the cohomology groups $H^k(\Gamma, \rho)$ were treated analytically and expressed as spaces of harmonic forms associated with the representation ρ.

In this paper, we shall deal purely algebraically with the Hecke operators on the cohomology groups $H^k(\Gamma, A)$ of arbitrary subgroups Γ of any abstract group G over a G-module A. The action of Hecke operators on $H^k(\Gamma, A)$, formulated by Kuga in [2] when G is a Lie group, turns out to be a sort of transfer map in the cohomology of groups.

In Section I, we described the Hecke rings $\mathfrak{R}(G, \Delta, \Gamma)$, and in Section II we obtained a representation of the Hecke rings $\mathfrak{R}(G, \Delta, \Gamma)$ over the cohomology groups $H^k(\Gamma, A)$ with an explicit formula. In the last section, we computed the effect of Hecke operators on $H^k(\Gamma, A)$ for a cyclic group Γ of $SL(2, \boldsymbol{Z}/p\boldsymbol{Z})$.

I. Hecke rings

1. Let G be a group. Two subgroups Γ and Γ' of G are said to be commensurable, denoted by $\Gamma \approx \Gamma'$, if the intersection of Γ and Γ' is of finite index with respect to both Γ and Γ'; in notation, $\Gamma \approx \Gamma' \Leftrightarrow [\Gamma : \Gamma \cap \Gamma'] < \infty$ and $[\Gamma' : \Gamma \cap \Gamma'] < \infty$. Then the commensurability is an equivalence relation and is invariant under conjugation, namely, $\Gamma \approx \Gamma'$ if and only if $\alpha^{-1}\Gamma\alpha = \Gamma^{\alpha} \approx \Gamma'^{\alpha}$. Let $\tilde{\Gamma}$ be the set of all elements α of G with $\Gamma^{\alpha} \approx \Gamma$.

PROPOSITION 1.1. *$\tilde{\Gamma}$ is a subgroup of G.*

PROOF. Given α and β in $\tilde{\Gamma}$, we have $\Gamma^{\alpha\beta} = (\alpha^{-1}\Gamma\alpha)^{\beta} \approx \Gamma^{\beta} \approx \Gamma$ and so $\alpha\beta$ belongs to $\tilde{\Gamma}$. By substituting α^{-1} for β, $\Gamma = (\alpha^{-1}\Gamma\alpha)^{\alpha^{-1}} \approx \Gamma^{\alpha^{-1}}$ implies $\alpha^{-1} \in \tilde{\Gamma}$.