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Deformations of compact complex surfaces II

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§ 0. Introduction.

This is a continuation of the paper [4], referred to as Part I in this paper. We employ the notation of Part I, in which we call connected compact complex manifolds of complex dimension 2 " surfaces ". The purpose of the present paper is to prove

THEOREM III. *Plurigenera of surfaces are invariant under arbitrary holomorphic deformations.*

The proof is based heavily on the classification theory of all surfaces established by Italian algebraic geometers and K. Kodaira [7].

§ 1. Proof for surfaces of which all plurigenera vanish.

First, we consider an algebraic surface which is birationally equivalent to the product of an algebraic curve and a projective line. We call this ruled surface, extending the previous definition of ruled surfaces in [7, IV].

PROPOSITION 1. *Any deformation of a ruled surface is also ruled.*

PROOF. For a rational surface, this is Theorem I in Part I, and for an irrational ruled surface this follows immediately from the classification of surfaces in [7, IV]. q. e. d.

By Proposition I and the vanishing of all plurigenera of a ruled surface, *Theorem III is proved when a ruled surface appears as a deformation of the surface.*

Second, we shall consider the other surfaces of which all plurigenera vanish.

PROPOSITION 2. *All plurigenera of surfaces with* $b_1=1$ *and* $P_{12}=0$ *vanish and the class consisting of such surfaces is closed under deformations.*

PROOF. The former part of Proposition 2 has been proved in Theorem 35 in [7, II]. To prove the latter part, we recall the following Lemma A, which was used in Part I.

LEMMA A. *Let* X *and* Y *be complex manifolds, and let* f *be a proper and simple holomorphic map from* X *to* Y, *such that every fibre* X_y *is a surface. The following two assertions can be established.*