J. Math. Soc. Japan
Vol. 22, No. 1, 1970

A theorem on the second-order arithmetic with the ω-rule*)

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(Received Dec. 3, 1968)

It is well known that the first order arithmetic with the ω-rule is complete. Moreover, J. R. Shoenfield [3] has shown that the same holds also when the ω-rule is recursively restricted. On the other hand, the second-order arithmetic with the ω-rule is not complete. So Shoenfield has raised a question whether every sentence of the second-order arithmetic provable with the ω-rule is provable with the recursively restricted ω-rule. Concerning this problem, H. Tanaka [4] has shown that every sentence of the second-order arithmetic provable with the ω-rule is provable with the hyper-arithmetically restricted ω-rule.

The purpose of this paper is to give an affirmative answer to Shoenfield's problem stated above. This result can be extended to one corresponding to any higher order arithmetic. We shall use notations and terminologies in [2].

Roughly speaking, the outline of proof is as follows. For a given formula φ, we define a tree T (consisting of formulas) whose only root is φ and which has the following properties: (i) whenever φ is a provable formula, the tree T is well-founded (in a suitable sense), and (ii) in T if $\psi_0, \psi_1, \cdots$ are direct predecessors of ψ then we can effectively construct a "recursive proof" of ψ from information for recursive proofs of ψ_i's. Then by means of Kleene's recursion theorem, there exists a partial recursive function π such that if $\psi, \psi_0, \psi_1, \cdots$ are as above-mentioned and if $\pi(\psi_i)$ is a recursive proof of ψ_i for each i, then $\pi(\psi)$ is a recursive proof of ψ. Thus, if φ is provable, then $\pi(\varphi)$ gives a recursive proof of φ, as is shown by the induction using the well-foundedness of T.

In what follows, we shall carry out a detailed proof based on this idea.

§1. As a formal system of second order arithmetic, we shall use A_ω in [1]. By a familiar way we can assign, to each formula φ of A_ω, a number

*) After the completion of this manuscript the author learned that Lopez-Escobar [5] proved the theorem for the case of the weak second-order logic (not the full system of second-order arithmetic) by a similar way as ours.