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Spectral synthesis for the Kronecker sets

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Throughout this paper, let G be any locally compact abelian group and $\hat{G}$ its dual. We denote by $A(G)$ the Banach algebra consisting of the Fourier transforms of all complex-valued functions on $\hat{G}$ that are absolutely summable with respect to the Haar measure of $\hat{G}$ [2].

N. Th. Varopoulos proved in [4] that every totally disconnected Kronecker subset of G is a set of spectral synthesis (an S-set) for the algebra $A(G)$. On the other hand, every compact (Hausdorff) space is homeomorphic to a Kronecker subset of some compact abelian group (see Theorem 2). The main purpose of this paper is to show that every Kronecker set is an S-set.

DEFINITION 1. A compact subset K of the group G is called a quasi-Kronecker set, provided that: For each $\varepsilon > 0$ and each real continuous function h on K ($h \in C_R(K)$), there exists a character $\gamma \in \hat{G}$ such that

$$\sup_{x \in K} | \exp [i\, h(x)] - (x, \gamma) | < \varepsilon .$$

It is then easy to see that:

(i) Every quasi-Kronecker set is independent;

(ii) A Kronecker set is a quasi-Kronecker set;

(iii) If K is a quasi-Kronecker subset of G, then we have $\|\mu\| = \|\hat{\mu}\|_\infty$ for all $\mu \in M(K)$. In particular, every quasi-Kronecker set is a Helson set.

The following theorem seems to be well-known. But the author does not know any literature about it; hence we give here a complete proof of it.

THEOREM 2. *There exists a compact abelian group which contains a quasi-Kronecker set that is not a Kronecker set. Every compact space is homeomorphic to a Kronecker subset of some compact abelian group.*

PROOF. Suppose that X is a compact space, and that a and b are two constants such that $0 < a < b < 1$, and take any subset F of $C_R(X)$ such that:

(2.1) We have $a \leqq f \leqq b$ for all $f \in F$;

(2.2) The functions in F separate points of X.

Let us then denote by $\mathfrak{F}$ the set of all functions in $C_R(X)$ expressible as a finite product of elements in F, and let