J. Math. Soc. Japan
Vol. 21, No. 4, 1969

On Stiefel manifolds

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(Received Feb. 24, 1969)
(Revised May 9, 1969)

Introduction.

T. T. Frankel [5] applied Morse theory to the Stiefel manifolds using the trace function. The critical sets in this case are Grassmann manifolds. In this note we apply Morse theory to the Stiefel manifolds using "length function". We think of Stiefel manifolds as imbedded in Euclidean spaces and use methods similar to R. Bott [3]. Finally, using some results on P. A. Smith theory of periodic transformations we show that the Morse inequalities are equalities. This method is due to Frankel [5]. The CW-decomposition and the Poincaré polynomials obtained for Stiefel manifolds are, of course, known. For this reason detailed proofs are omitted.

The referee points out that "the length function" is essentially the same as the function used by Takeuchi and Kobayashi [7] generalizing the trace function of Frankel [5]. The author is grateful to the referee for this and other valuable suggestions and comments.

Preliminaries.

Let F be R, the field of real numbers, C the field of complex numbers or Q, the quaternions. Let $U(n\,;F)=\{A\,|\,A\bar{A}^t=I_n\}$ where A is an $n\times n$ matrix with coefficients in F. The 'bar' denotes complex conjugation or the quaternionic conjugation as the case may be. Let $U_0(n\,;F)$ be the identity component of $U(n\,;F)$. Hence $U_0(n\,;F)$ is $SO(n)$ if $F=R$, is $U(n)$ if $F=C$, and is $Sp(n)$ if $F=Q$. Let $\underline{u}(n\,;F)$ be the Lie algebra of $U(n\,;F)$. Let $V_{p+q,p}(F)=\dfrac{U_0(p+q\,;F)}{U_0(q\,;F)}$ be Stiefel manifold over F. If $q=0$, we get the classical groups; $V_{p+q,p}(F)$ is the set of all orthogonal p-frames in F^{p+q} space with respect to the standard metric $\sum x_i\bar{x}_i$.

The Stiefel manifolds are imbedded in Euclidean spaces as follows: Let G be a compact connected Lie group with an invariant Riemannian metric. (We will take G to be $U_0(n\,;F)$). Let $\sigma: G\rightarrow G$ be an involution with the full

* This research was supported in part by NSF Grant GP-5874.