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Interpolation by the real method preserves compactness of operators

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In this paper we will prove the following

THEOREM. *Let* $[E_0, E_1]$ *and* $[F_0, F_1]$ *be arbitrary interpolation pairs, and let* T *be a continuous linear operator from the couple* $[E_0, E_1]$ *to the couple* $[F_0, F_1]$. *If the mappings* $T: E_0 \to F_0$ *and* $T: E_1 \to F_1$ *are compact, then for* $1 \leqq p < \infty$, $0 < \theta < 1$ $T: S(\theta, p; E_0, E_1) \to S(\theta, p; F_0, F_1)$ *is compact. Here* $S(\theta, p; E_0, E_1)$ *is the interpolation space by the real method of Lions and Peetre* [**1**].

When the couple $[F_0, F_1]$ satisfies a certain approximation hypothesis, A. Persson [**3**] proved that if $T: E_0 \to F_0$ is compact, then $T: E_\theta \to F_\theta$ is also compact, where E_θ and F_θ are the interpolation spaces by the real or the complex method.

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§ 1. Notations, definitions and fundamental facts.

For two linear topological spaces $\mathcal{E}$ and $\mathcal{F}$, we write $\mathcal{E} \subset \mathcal{F}$ if $\mathcal{E}$ is a linear subspace of $\mathcal{F}$ and the identity map is continuous.

A pair of Banach spaces $[E_0, E_1]$ is said to be an interpolation pair if there exists a Hausdorff linear topological space $\mathcal{E}$ such that $E_0 \subset \mathcal{E}$ and $E_1 \subset \mathcal{E}$. In this paper, when we write $[E_0, E_1]$ or $[F_0, F_1]$ we always assume that the pair is an interpolation pair.

For $[E_0, E_1]$ we can define Banach spaces $E_0 \cap E_1$ and $E_0 + E_1$ with norms

$$\|x\|_{E_0 \cap E_1} = \mathrm{Max}\,(\|x\|_{E_0},\ \|x\|_{E_1}),$$

and

$$\|x\|_{E_0 + E_1} = \mathrm{Inf}\,(\|x_0\|_{E_0} + \|x_1\|_{E_1};\ x = x_0 + x_1)$$

respectively.

Given a Banach space E and real numbers p and θ $(1 \leqq p \leqq \infty)$, we consider E-valued sequences $\{a_m\}_{m=-\infty}^{\infty}$ such that $\{e^{m\theta}\|a_m\|_E\} \in l^p$. In the linear space of all those sequences, which is denoted by $l_\theta^p(E)$, we introduce the norm