J. Math. Soc. Japan
Vol. 20, No. 4, 1968

An investigation on degrees of unsolvability

Dedicated to Professor Motokiti Kondô on his sixtieth birthday anniversary

By Ken HIROSE

(Received Nov. 29, 1965)

§ 0. Introduction.

By *degree*, we mean the degree of recursive unsolvability as defined by S. C. Kleene and E. L. Post in [2]. For notations not explained here, see [1], [2] and [5].

For each degree $\boldsymbol{d}$, let $R_{\boldsymbol{d}}$ denote the set of all degrees greater than or equal to $\boldsymbol{d}$, recursively enumerable in $\boldsymbol{d}$ and less than or equal to $\boldsymbol{d}'$ (the completion of $\boldsymbol{d}$).

R. M. Friedberg has shown that degree $\boldsymbol{d}'$ does not have a unique pre-image in $R_{\boldsymbol{d}}$. G. E. Sacks [4] proved that if $\boldsymbol{a} \in R_{\boldsymbol{b}'}$, then there exists a degree $\boldsymbol{c}$ such that $\boldsymbol{c} \in R_{\boldsymbol{b}}$ and $\boldsymbol{c}' = \boldsymbol{a}$.

The main result of the present paper is that if $\boldsymbol{a} \in R_{\boldsymbol{b}'}$, then for any positive integer n, there exist independent degrees $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n$ such that $\boldsymbol{c}_i \in R_{\boldsymbol{b}}$ and $\boldsymbol{c}_i' = \boldsymbol{a}$ for $i = 1, 2, \cdots, n$. Thus the degrees which lie between $\boldsymbol{b}'$ and $\boldsymbol{b}''$ and are recursively enumerable in $\boldsymbol{b}'$ can be viewed as the completions of the independent degrees which lie between $\boldsymbol{b}$ and $\boldsymbol{b}'$ and are recursively enumerable in $\boldsymbol{b}$. This shall be proved as a corollary of the following 'main theorem'. The methods used here are those developed in [2], [3] and [4].

We shall denote by $\boldsymbol{a} \upharpoonleft \boldsymbol{b}$ the relation between degrees $\boldsymbol{a}$ and $\boldsymbol{b}$: $\boldsymbol{a}$ is recursively enumerable in $\boldsymbol{b}$.

MAIN THEOREM. *Let* $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *be degrees such that:*

(I) $\boldsymbol{a} \not\leqq \boldsymbol{b}$

(II) $\boldsymbol{a} \leqq \boldsymbol{b}' \leqq \boldsymbol{c}$

(III) $\boldsymbol{c} \upharpoonleft \boldsymbol{b}'$

Then for any positive integer n, *there exist degrees* $\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{n-1}$ *such that:*

(i) $\boldsymbol{b} \leqq \boldsymbol{d}_i$ *for* $i = 0, 1, \cdots, n-1$,

(ii) $\boldsymbol{d}_i \upharpoonleft \boldsymbol{b}$ *for* $i = 0, 1, \cdots, n-1$,

(iii) $\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{n-1}$ *are independent,*

(iv) $\boldsymbol{a} \not\leqq \boldsymbol{d}_i$ *for* $i = 0, 1, \cdots, n-1$,

(v) $\boldsymbol{d}_i' = \boldsymbol{c}$ *for* $i = 0, 1, \cdots, n-1$.