J. Math. Soc. Japan
Vol. 20, No. 3, 1968

On provably recursive functions and ordinal recursive functions*

By Akiko KINO

(Received Oct. 24, 1966)

A recursive function $\phi(x)$ is defined to be $U(\mu y T(e, x, y))$, if $\forall x \exists y T(e, x, y)$, where U and T are primitive recursive and e is an integer; but nothing is said about the theory in which the predicate $\forall x \exists y T(\mathbf{e}, x, y)$ is provable. The investigation of reasonable theories $\mathfrak{T}$ in which provable recursiveness in $\mathfrak{T}$ is defined by $\vdash_{\mathfrak{T}} \forall x \exists y T(\mathbf{e}, x, y)$ forms an interesting branch of recursive function theory, and the functions provably recursive in such $\mathfrak{T}$ constitute a not unnatural subclass of the class of computable functions. We will give a characterization of provable recursiveness for certain theories.

Let $\mathfrak{T}$ be the theory of natural numbers or a subtheory of analysis. A recursive function $\phi(x)$ is called "provably recursive in $\mathfrak{T}$", if $\vdash_{\mathfrak{T}} \forall x \exists y T(\mathbf{e}, x, y)$, where e is a Gödel number of ϕ. Let $\prec$ be a primitive recursive well-ordering of natural numbers with $\neg n' \prec 0$ for every n. We call $\prec$ a *provable primitive recursive well-ordering in* $\mathfrak{T}$, if the sentence "$\prec$ is a well-ordering" is provable in $\mathfrak{T}$ (cf. § 3). A number-theoretic function ϕ is called "ordinal recursive with respect to $\prec$" ($\prec$-recursive), if it is defined by "defining equations" of primitive recursive form and by transfinite induction with respect to $\prec$. (For the precise definition, cf. [**8a**] and § 2.)

In [**11**], Takeuti defined *GLC*, a Gentzen-style simple type theory containing t-variables of the first order and f-variables with finitely many argument-places and stated his fundamental conjecture (FC) about *GLC*; (that Gentzen's Hauptsatz for *LK*, that is the cut elimination theorem, holds in *GLC* as well.) Takeuti proved that FC holds for many subsystems of *GLC* by using transfinite

---

* The author wishes to express her heartfelt thanks to Professor G. Takeuti for his invaluable advice during the preparation of this paper. This work was done at Hughes Aircraft Company, Fullerton, California, and was sponsored by Air Force Systems Command, Research and Technology Division, Rome Air Development Center, Griffiss Air Force Base, New York, 13442, under Contract AF 30(602)-3754. An earlier version was read at the RADC-HAC Joint Symposium on Logic. Computability and Automata held at Oriskany, N.Y. in August 1965. The author is indebted to Professor J. Myhill, Drs. F.B. Cannonito and V.H. Dyson, and Mr. G.E. Cash for reading this paper in manuscript and suggesting a number of linguistic improvements.