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A type of integral extetsions

To Professor Iyanaga for celebration of his 60th birthday

By Masayoshi NAGATA

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The purpose of the present paper is to prove the following

THEOREM. *Let* $S \subsetneqq R$ *be integral domains with fields of quotients* $Q(S) \subsetneqq Q(R)$. *Assume that for each element* r *of* R, *there is a natural number* n *(depending on* r*) such that* r^n *is in* $Q(S)$. *Then either* (1) $Q(R)$ *is purely inseparable over* $Q(S)$ *or* (2) R *and* S *are algebraic over a finite field.*

The proof is given as follows. Assume that $Q(R)$ is not purely inseparable over $Q(S)$. Then there is an element a of R which is not in $Q(S)$ and which is separable over $Q(S)$. We fix this element a. Let $a=a_1, a_2, \cdots, a_c$ be all of the conjugates of a over $Q(S)$ in an algebraically closed field K containing $Q(R)$. If S contains only a finite number of elements, then (2) holds good obviously. Therefore we assume that S contains infinitely many elements. For each element s of S, there is a natural number $n(s)$ such that $(a+s)^{n(s)} \in Q(S)$ and such that $(a+s)^m \notin Q(S)$ for every natural number m which is less than $n(s)$.

Case 1. Assume that there is an infinite subset S^* of S such that $\{n(s) \mid s \in S^*\}$ is bounded. In this case, there is a natural number N such that $n(s)=N$ for an infinite subset S^{**} of S^*. Take mutually distinct elements, $s_0, s_1, \cdots, s_N$ from S^{**} and consider the relations

$$a^N+\binom{N}{1}s_i a^{N-1}+\cdots+\binom{N}{\alpha}s_i^{\alpha}a^{N-\alpha}+\cdots+s_i^N=b_i \in Q(S)$$

$$(i=0, 1, \cdots, N).$$

Since the matrix

$$A=\begin{pmatrix} 1 & s_0 & \cdots & s_0^N \\ 1 & s_1 & \cdots & s_1^N \\ & \cdots\cdots & & \\ & \cdots\cdots & & \\ 1 & s_N & \cdots & s_N^N \end{pmatrix}$$

is non-singular, we see that the non-zero columns in