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On the unit group of an absolutely cyclic number field of degree five

Dedicated to Professor Iyanaga on his 60th birthday

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1. Let K be a Galois extension of odd degree n over the rational number field $\boldsymbol{Q}$. Then K is totally real and the group of units of K has $(n-1)$ generators mod ± 1. Let $\boldsymbol{H}$ be the group of totally positive units of K. Then $\boldsymbol{H}$ has also $(n-1)$ generators, and it is known that in case $n=3$ these generators can be taken to conjugate to each other (cf. Hasse [1]). We shall show in this paper that the same is true for $n=5$.

In the following let K be a cyclic field of degree 5 over $\boldsymbol{Q}$, σ a generator of the Galois group $G(K/\boldsymbol{Q})$ and $\boldsymbol{H}$ the group of totally positive units of K. For $\xi \in K$, $\xi^{(i)}$ means $\sigma^{i-1}(\xi) \in K$ $(i=1, 2, 3, 4, 5)$. Then the points

$$\boldsymbol{P}(\xi)=(\log \xi^{(1)}, \log \xi^{(2)}, \log \xi^{(3)}, \log \xi^{(4)}, \log \xi^{(5)}) \in \boldsymbol{R}^5$$

for $\xi \in \boldsymbol{H}$ form a lattice $\boldsymbol{L}$ lying in the hyperplane $\pi: x_1+x_2+x_3+x_4+x_5=0$. Obviously the five points $\boldsymbol{P}(\xi^{(1)}, \cdots, \boldsymbol{P}(\xi^{(5)})$ lie at the same distance from the origin O of $\boldsymbol{R}^5$.

Let $\eta(\neq 1)$ be a unit in $\boldsymbol{H}$ such that $\boldsymbol{P}(\eta) \in \boldsymbol{L}$ lies nearest to O. Then our main result is that $\boldsymbol{H}$ is generated by any four of $\eta^{(1)}, \eta^{(2)}, \eta^{(3)}, \eta^{(4)}, \eta^{(5)}$, or geometrically expressed, $\boldsymbol{L}$ is generated by $\boldsymbol{P}(\eta^{(1)}), \cdots, \boldsymbol{P}(\eta^{(5)})$.

We shall namely prove the following theorem.

THEOREM. *Let K be an absolutely cyclic field of degree* 5, *and $\boldsymbol{H}$ the group of totally positive units of K. Then $\boldsymbol{H}$ is generated by $\eta \in \boldsymbol{H}$ and its conjugates, where η is an element* $(\neq 1)$ *of $\boldsymbol{H}$ such that*

$$\sum_{i=1}^{5} (\log \eta^{(i)})^2 \leqq \sum_{i=1}^{5} (\log \xi^{(i)})^2$$

holds for any element $\xi \in \boldsymbol{H}$ $(\xi \neq 1)$.

2. We shall first prove the following general proposition. Let $\boldsymbol{M}$ be an n-dimensional lattice in $\boldsymbol{R}^n$, which is generated by n vectors $\overrightarrow{OQ}_1, \overrightarrow{OQ}_2, \cdots, \overrightarrow{OQ}_n$. Let d_i be the length of $\overrightarrow{OQ}_i$ $(i=1, 2, \cdots, n)$.