J. Math. Soc. Japan
Vol. 20, Nos. 1-2, 1968

The canonical modification of stochastic processes

Dedicated to Professor S. Iyanaga for his sixtieth birthday

By Kiyosi ITÔ

(Received Sept. 11, 1967)

§1. Introduction and summary.

In his book [1], J. L. Doob proved that every stochastic process continuous in probability has a standard separable measurable modification. This theorem plays a fundamental role in the sample path approach of stochastic processes. The aim of our paper is to give a more concrete formulation to this important fact to make it easier to visualize the probability law of the sample path.

In Section 2 we shall introduce the space $M \equiv M(T)$ of *canonical measurable functions* on the time interval T. The space $\tilde{M} \equiv \tilde{M}(T)$ contains bad functions such as the Dirichlet function that takes 1 on rationals and 0 elsewhere. Since we have a good function $f \equiv 0$ equivalent to the Dirichlet function, this can be discarded from $\tilde{M}$ without any essential loss. We shall pick up at least one good function, called canonical measurable function here, from among each equivalent class in $\tilde{M}$ and consider the space $M \equiv M(T)$ of all canonical functions in behalf of $\tilde{M}$. By definition a canonical function takes one of its general approximate limits at each point. All continuous functions are in M and if a function in M is equal to a continuous function almost everywhere on T, they are equal everywhere on T. A similar fact holds for functions with no discontinuities of the second kind. These facts suggest that M is suitable for the function space in which the path of a reasonable stochastic process is ranging.

In Section 3 we shall define a σ-algebra $\mathcal{B} = \mathcal{B}(M)$ of subsets of M which will determine a measurable structure in M. $\mathcal{B}$ is generated by all sets of the following types

$$\text{(i)} \qquad \{f \in M : f(t) < a\}\,,$$

$$\text{(iii)} \qquad \{f \in M : \int_I \arctan f(t)dt < a\}\,.$$

where a ranges over reals, t over T and I over all compact intervals in T. The scaling "arctan" was used in the integral to make it converge. We shall write $\mathcal{B}_K \equiv \mathcal{B}_K(M)$ and $\mathcal{B}_\rho \equiv \mathcal{B}_\rho(M)$, respectively, for the σ-algebra generated