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On rational points of homogeneous spaces over finite fields

Dedicated to Professor S. Iyanaga on his 60th birthday

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Let G be a connected algebraic group and V a homogeneous space for G, which are defined over a finite field k. We denote by G_k the subgroup of G consisting of all the rational points over k and also by V_k the subset of V consisting of all the rational points over k. Then the operation of G to V induces an operation of G_k to V_k and so V_k is considered as a transformation space for G_k in the abstract sense.

The purpose of this paper is to calculate the number of the G_k-orbits in V_k and the number of points in each G_k-orbit, under an assumption on k, which will be referred to by $(*)$[1]. The main results are as follows (under the assumption $(*)$):

1) Let P_0 be a point in V_k and H the isotropy group of P_0 in G. Let s be the number of conjugate classes of the finite group H/H_0[2]. Then V_k is decomposed into the disjoint union of s G_k-orbits (Theorem 1). This fact is a consequence of 'Galois cohomology theory' (cf. [7]), but we shall give here an elementary proof of it. On the other hand, we can give an example, which shows that the number of points of each G_k-orbit is not necessarily same to each other.

2) We restrict ourselves to the case where V is complete. Then it is proved that H/H_0 is commutative and the normalizer $N(H)$ of H in G is connected (Proposition 1). From these facts, we can show that the number of G_k-orbits in V_k is equal to the index $(H:H_0)$ and the numbers of points in any G_k-orbits are all same (Theorem 2). Moreover, if G operates effectively on V, it is also proved that H is connected (Proposition 2). Hence, in this case, we see that V_k is a homogeneous space for G_k in the abstract sense (Theorem 2′).

3) Let $\mathfrak{g}$ be a finite subgroup of G_k. Then, we shall prove that the num-

1) Cf. the beginning of the section 2.

2) For an algebraic group H, we denote by H_0 the connected component containing the identity element.