J. Math. Soc. Japan
Vol. 20, Nos. 1-2, 1968

A new proof of the Baker-Campbell-Hausdorff formula

Dedicated to Professor Shôkichi Iyanaga on his 60th birthday

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(Received Jan. 11, 1967)

This formula states

$$e^{A}\cdot e^{B}=e^{Z}, \qquad Z=\sum_{n=1}^{\infty}F_{n}(A, B) \tag{1}$$

for noncommuting indeterminates A, B with homogeneous polynomials $F_{n}(A, B)$ of degree n which have the essential property that they are formed from A, B by Lie multiplication, except for $F_{1}(A, B)=A+B$. We shall briefly speak of Lie polynomials. The usual proofs (e. g. [**1**], [**2**]) employ preliminary theorems by Finkelstein or Friedrichs characterizing Lie polynomials by formal properties (see also [**3**]). In the following lines I give a short proof which needs no preparations.

It is evident that polynomials $F_{n}(A, B)$ exist satisfying (1). We only have to prove that they are Lie polynomials. The first two are

$$F_{1}(A, B)=A+B, \qquad F_{2}(A, B)=\frac{1}{2}(AB-BA).$$

Now let $n>2$ and assume that all $F_{\nu}(A, B)$ with $\nu<n$ are Lie polynomials. With 3 indeterminates we express

$$(e^{A}e^{B})e^{C}=e^{A}(e^{B}e^{C}):$$

$$W=\sum_{i=1}^{\infty}F_{i}\Big(\sum_{j=1}^{\infty}F_{j}(A, B), C\Big)=\sum_{i=1}^{\infty}F_{i}\Big(A, \sum_{j=1}^{\infty}F_{j}(B, C)\Big)$$

and compare the homogeneous terms of degree n on both sides, using the following 2 facts: 1) If $F(A, B, \cdots)$, $X(A, B, \cdots)$, $Y(A, B, \cdots), \cdots$ are Lie polynomials then also $G(A, B, \cdots)=F(X(A, B, \cdots), Y(A, B, \cdots), \cdots)$ is one. 2) If $F(A, B, \cdots)$ is a Lie polynomial then the homogeneous summands into which F splits up are Lie polynomials. The induction assumption implies that all homogeneous terms of degree n in both expressions for W are Lie polynomials with the possible exceptions of $F_{n}(A, B)+F_{n}(A+B, C)$ on the left side and $F_{n}(A, B+C)+F_{n}(B, C)$ on the right. In other words, the difference is a Lie polynomial. We can abbreviate this as

$$F(A, B)+F(A+B, C)\sim F(A, B+C)+F(B, C) \tag{2}$$