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Invariant distances on complex manifolds and holomorphic mappings

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1. Introduction

The classical Schwarz lemma, in its invariant form formulated by Pick, states that every holomorphic mapping of the unit disk into itself is distance-decreasing with respect to the Poincaré-Bergman metric. It has been since generalized to higher dimensions in various forms, [**1, 2, 10, 12, 16, 17, 20, 21** etc.]. Most of these generalizations originate from Ahlfors's generalization of Schwarz lemma [**1**]. The essence of these generalizations is that, given complex manifolds M and N endowed with either metrics or volume elements, every holomorphic mapping $f: M \to N$ is distance- or volume-decreasing under the conditions that M is a ball or a symmetric domain in $\boldsymbol{C}^m$ and that N has negative curvature in one sense or other. In this way we get some control over the family of holomorphic mappings $f: M \to N$.

In [**18**] I announced a canonical way of constructing a new pseudo-distance d_M on each complex manifold M. In this paper we prove basic properties of these pseudo-distances and apply them to the study of holomorphic mappings. The two most important properties of d_M are that every holomorphic mapping $f: M \to N$ is distance-decreasing with respect to d_M and d_N and that these pseudo-distances are often true distances. For instance, if M is a Riemann surface covered by a disk or more generally a complex manifold covered by a bounded domain, then d_M is a true distance. We call a complex manifold *hyperbolic* if its invariant pseudo-distance is a distance. If N is hyperbolic, we can therefore draw useful conclusion on the value distribution of a holomorphic mapping $f: M \to N$. The first basic question in the study of holomorphic mappings from this view point is therefore to decide whether the image manifold is hyperbolic and also complete with respect to its invariant distance. From Ahlfor's generalized Schwarz lemma we can prove that if a complex manifold admits a (complete) hermitian metric of strongly negative curvature or a (complete) differential metric of strongly negative curvature in the sense of Grauert-Reckziegel, then it is (complete) hyperbolic. But there

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